Note] $V\left(\sum_{i=1}^{n} Y_i\right) = cov\left(\sum_i Y_i, \sum_j Y_j\right)$

$= \sum_i \sum_j cov(Y_i, Y_j) = \sum_i \sum_j \sigma_{ij}$

terms in matrix sum to $\sum_i \sum_j \sigma_{ij}$

symmetric $\rightarrow \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & & \sigma_{2n} \\ \vdots & & & \\ \sigma_{n1} & \sigma_{n2} & & \sigma_{nn} \end{bmatrix}$

main diagonal sums to $\sum_i V(Y_i)$

entries above main diagonal sum to $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sigma_{ij}$

So $V\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} V(Y_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} cov(Y_i, Y_j)$.

54] $\sum_{i=1}^{n} a_i Y_i$ and $\sum_{i=1}^{n} a_i t(Y_i)$ are important statistics. often $a_i \equiv 1$ or $a_i \equiv \frac{1}{n}$.

55] If W and Z have the same mgf $m(t)$ for $|t| < t_0$ then $W \sim Z$.

56] Know for E2 If $X_1, \ldots, X_n$ are ind and $W = X_1 + \cdots + X_n = \sum_{i=1}^{n} X_i$, then the mgf of W

is $m_W(t) = \prod_{i=1}^{n} m_{X_i}(t) = m_{X_1}(t) \cdots m_{X_n}(t)$

Proof] $m_W(t) = E e^{Wt} = E e^{(\sum X_i)t} = E e^{X_1 t + \cdots + X_n t} = \prod_{i=1}^{n} \underbrace{E e^{X_i t}}_{m_{X_i}(t)}$

ex] see ex 2.20

ex] #2.6 X_i ind gamma (ν_i, λ) $W = \sum_{i=1}^{n} X_i$

$m_W(t) = \prod_{i=1}^{n} \left(\frac{1}{1-\lambda t}\right)^{\nu_i} = \left[\frac{1}{1-\lambda t}\right]^{\sum_{i=1}^{n} \nu_i}$

so $W \sim G\left(\sum_{i=1}^{n} \nu_i, \lambda\right)$

ex] # 2.4 X_i ind $N(\mu_i, \sigma_i^2)$, $W = \sum_{i=1}^n X_i$

$$m_W(t) = \prod_{i=1}^n e^{\mu_i t + \sigma_i^2 t^2/2} = e^{\left(\sum_{i=1}^n \mu_i\right)t + \left(\sum_{i=1}^n \sigma_i^2\right)t^2/2}$$

So $W \sim N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$.

If $\mu_i \equiv \mu$ and $\sigma_i^2 \equiv \sigma^2$ then $W \sim N(n\mu, n\sigma^2)$.

57] Theorem 2.17 has important examples of this type.

§2.7 58] $\underset{\sim}{Y} = (Y_1, \ldots, Y_p)$ is a random vector.

59] p57 $E(\underset{\sim}{Y}) = (E(Y_1), \ldots, E(Y_p))$.

60] If $\underset{\sim}{Y}$ and $\underset{\sim}{Y}^T$ are used, then

$\underset{\sim}{Y}$ is a column vector $(p \times 1)$ and
$\underset{\sim}{Y}^T$ is a row vector $(1 \times p)$.

61] Let $\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_p \end{pmatrix}$, Then $\text{Cov}(\underset{\sim}{Y}) =$

$$E(\underset{\sim}{Y} - E(\underset{\sim}{Y}))(\underset{\sim}{Y} - E(\underset{\sim}{Y}))^T = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \sigma_{21} & \cdots & \sigma_{2p} \\ \vdots & & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{pmatrix}$$

where $\sigma_{ij} = \text{Cov}(Y_i, Y_j)$.

62] p58 Let $\underset{\sim}{X}$, $\underset{\sim}{Y}$ be $p \times 1$, $\underset{\sim}{a}$ a conformable constant vector, let A and B be conformable constant matrices.

$E(\underset{\sim}{a} + \underset{\sim}{X}) = \underset{\sim}{a} + E(\underset{\sim}{X})$, $E(\underset{\sim}{X} + \underset{\sim}{Y}) = E(\underset{\sim}{X}) + E(\underset{\sim}{Y})$

$\text{Cov}(\underset{\sim}{a} + A\underset{\sim}{X}) = \text{Cov}(A\underset{\sim}{X}) = A \,\text{Cov}(\underset{\sim}{X})\, A^T$.

63] p58 Let $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n$ be random vectors with joint pdf or pmf $f(\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_n)$.

$\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n$ are ind if $f(\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_n) = \prod_{i=1}^{n} f_{\underset{\sim}{Y}_i}(\underset{\sim}{y}_i)$.

64] p58 If $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n$ are ind with $\underset{\sim}{Y}_i$ $1 \times p_i$, then $\underset{\sim}{h}_1(\underset{\sim}{Y}_1), \ldots, \underset{\sim}{h}_n(\underset{\sim}{Y}_n)$ are ind where

$\underset{\sim}{h}_i : \mathbb{R}^{p_i} \to \mathbb{R}^{p_{j_i}}$ (ie dimension of vector $\underset{\sim}{h}_i$ is j_i & depends on i)

65] The mgf $m_{\underset{\sim}{Y}}(\underset{\sim}{t}) = E(e^{\underset{\sim}{t}^T \underset{\sim}{Y}}) = E[e^{t_1 Y_1 + \cdots + t_p Y_p}]$

if the expectation exists for $\|\underset{\sim}{t}\| \le t_0$.

66] $$E(Y_{i_1}^{k_1} \cdots Y_{i_j}^{k_j}) = \frac{\partial^{k_1 + \cdots + k_j}}{\partial t_{i_1}^{k_1} \cdots \partial t_{i_j}^{k_j}} m(\underset{\sim}{t}) \Big|_{\underset{\sim}{t} = \underset{\sim}{0}}$$

so $$E(Y_i) = \frac{\partial m(\underset{\sim}{t})}{\partial t_i}\Big|_{\underset{\sim}{t} = \underset{\sim}{0}}$$ and

$$E(Y_i Y_j) = \frac{\partial^2 m(\underset{\sim}{t})}{\partial t_i \, \partial t_j}\Big|_{\underset{\sim}{t} = \underset{\sim}{0}}$$

67] Th 2.21 If $Y_1, \ldots, Y_n$ have mgf $m_{\underset{\sim}{Y}}(\underset{\sim}{t})$

then the mgf for $Y_{i_1}, \ldots, Y_{i_k}$ is found by replacing t_{i_j} by 0 for $j = k+1, \ldots, n$. So if $\underset{\sim}{t} = (\underset{\sim}{t}_1, \underset{\sim}{t}_2)$ and $\underset{\sim}{Y} = (\underset{\sim}{Y}_1, \underset{\sim}{Y}_2)$, then $m_{\underset{\sim}{Y}_1}(\underset{\sim}{t}_1) = m_{\underset{\sim}{Y}}(\underset{\sim}{t}_1, \underset{\sim}{0})$.

15.5

68] Th 2.22 $\underset{\sim}{Y} = (\underset{\sim}{Y}_1, \underset{\sim}{Y}_2)$. If the mgfs exist, $\underset{\sim}{Y}_1 \perp\!\!\!\perp \underset{\sim}{Y}_2$

iff $m_{\underline{Y}}(\underline{t}) = m_{\underline{Y}_1}(\underline{t}_1)\, m_{\underline{Y}_2}(\underline{t}_2) \quad \forall\ \underline{t} = (\underline{t}_1, \underline{t}_2)$

$\ni \|\underline{t}\| \le t_0$.

§2.8 69] p 60 Idea: m iid trials with n outcomes

Y_i = # of the m trials resulting in outcome i, $i = 1, \ldots, n$.

$0 \le \rho_i \le 1 \qquad \sum_{i=1}^{n} \rho_i = 1 \qquad \rho_i$ = prob of ith outcome

Then $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)$ has a multinomial $M_n(m, \rho_1, \ldots, \rho_n)$ distribution

if $f_{\underset{\sim}{Y}}(y_1, \ldots, y_n) = P(Y_1 = y_1, \ldots, Y_n = y_n)$

$$= \frac{m!}{y_1! \cdots y_n!} \rho_1^{y_1} \cdots \rho_n^{y_n} = m! \prod_{i=1}^{n} \frac{\rho_i^{y_i}}{y_i!}.$$

$$\mathcal{Y} = \{ \underline{y} \mid \sum_{i=1}^{n} y_i = m, \quad 0 \le y_i \le m, \ i = 1, \ldots, n \}.$$

70] p60 Multinomial theorem: Let m and n be positive integers and $x_1, \ldots, x_n$ any real numbers. Then $(x_1 + \cdots + x_n)^m = \sum_{\underset{\sim}{y} \in \mathcal{Y}} \frac{m!}{y_1! \cdots y_n!} x_1^{y_1} \cdots x_n^{y_n}$.

take $0^0 = 1$

Take $x_i = \rho_i$ to show that $f_{\underset{\sim}{Y}}$ is a joint pmf.

71] If $\underset{\sim}{Y} \sim M_n(m, \rho_1, \ldots, \rho_n)$, then $Y_i \sim \text{bin}(m, \rho_i)$

(m trials Y_i are successes, $P(S) = \rho_i$, $m - Y_i$ trials are Fs)

72] Let $Y_K^* = \underbrace{m - \sum_{j=1}^{K-1} Y_{i_j}}$ where $1 \le i_1 < i_2 < \dots < i_{K-1} \le n$.

collapse $n-k+1$ categories into category Y_K^*

like a marginal where $k-1$ categories $i_1, \dots, i_{k-1}$ are of interest, but collapse the other categories into a nuisance category

Then $(Y_{i_1}, \dots, Y_{i_{K-1}}, Y_K^*) \sim M_K(m, \rho_{i_1}, \dots, \rho_{i_{K-1}}, 1-\sum_{j=1}^{K-1} \rho_{i_j})$

73] p60 (In words t outcomes left to distribute among k outcomes, so multinomial. The prob of ρ_{i_j} but the $\sum_j \rho_{i_j} \ne 1$ so need to scale.

If $\underset{\sim}{Y} \sim M_n(m, \rho_1, \dots, \rho_n)$ and $t = m - \sum_{j=k+1}^{n} Y_{i_j}$,

then $(Y_{i_1}, \dots, Y_{i_K} \mid Y_{i_{K+1}} = y_{i_{K+1}}, \dots, Y_{i_n} = y_{i_n})$

$\sim M_K(t, \pi_{i_1}, \dots, \pi_{i_K})$ where $\pi_{i_j} = \dfrac{\rho_{i_j}}{\sum_{j=1}^{K} \rho_{i_j}}$

for $j = 1, \dots, K$.

74] $(Y_1, Y_2) \sim M_2(m, \rho, 1-\rho) \Rightarrow Y_1 \sim bin(m, \rho)$

75] p61 $\underset{\sim}{Y} \sim M_n(m, \rho_1, \dots, \rho_n) \Rightarrow E\, Y_i = m\rho_i$,

$V(Y_i) = m\rho_i(1-\rho_i)$ and $Cov(Y_i, Y_j) = -m\rho_i \rho_j$, $i \ne j$.

§2.9 76] A $p \times 1$ random vector $\underset{\sim}{X}$ has a <u>p-dimensional multivariate normal (MVN) distribution</u>

$\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$ iff $\underset{\sim}{t}^T \underset{\sim}{X}$ has a univariate normal distribution for any $p \times 1$ vector $\underset{\sim}{t}$.

$E(\underset{\sim}{X}) = \underset{\sim}{\mu}$, $cov(\underset{\sim}{X}) = \Sigma$.

77] p61 Usually want Σ to be positive definite.

Then $\underset{\sim}{X}$ has pdf $f(\underset{\sim}{z}) = \dfrac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp[-\frac{1}{2}(\underset{\sim}{z}-\underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{z}-\underset{\sim}{\mu})]$.

where $|\Sigma| = \det(\Sigma)$.

ex] If $p=1$, $\Sigma = \sigma^2$ and $(\underset{\sim}{z}-\underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{z}-\underset{\sim}{\mu}) = \frac{(z-\mu)^2}{\sigma^2}$

78] If $\underset{\sim}{X} = (X_1, \ldots, X_p)^T$ where the X_i are ind $N(\mu_i, \sigma_i^2)$, then $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$

where $\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$ and $\Sigma = \operatorname{diag}(\sigma_i^2)$.

79] * If $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$ and A is a $q \times p$ constant matrix, then $A\underset{\sim}{X} \sim N_q(A\underset{\sim}{\mu}, A\Sigma A^T)$.

80] p 62 If $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$, then all subsets are MVN $(X_{k_1}, \ldots, X_{k_q})^T \sim N_q(\underset{\sim}{\tilde{\mu}}, \tilde{\Sigma})$

where $\tilde{\mu}_i = E(X_{k_i})$ and $\tilde{\Sigma}_{ij} = \operatorname{cov}(X_{k_i}, X_{k_j})$.

ex] Know for E2 $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left[\begin{pmatrix} 1 \\ 17 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right]$.

Find the dist of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ and $\begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$.

Soln] $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 1 \\ 17 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\right]$

$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\right]$

$\begin{pmatrix} X_2 \\ X_3 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 17 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right]$

81] Let $\underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \end{pmatrix}\begin{matrix} q\times 1 \\ (p-q)\times 1 \end{matrix}$, $\underset{\sim}{\mu} = \begin{pmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$.