82) * If $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$, then

$\underset{\sim}{X}_1 \sim N_q(\underset{\sim}{\mu}_1, \Sigma_{11})$ and $\underset{\sim}{X}_2 \sim N_{p-q}(\underset{\sim}{\mu}_2, \Sigma_{22})$

83) * If $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$, then $\underset{\sim}{X}_1 \perp\!\!\!\perp \underset{\sim}{X}_2$ iff $\Sigma_{12} = 0$.

matrix of 0's

84] * If the joint dist of $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ is not MVN, $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ could be MVN, uncorrelated, but not independent.

85] The population correlation between 2 RVs X and Y is $\rho(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\sqrt{V(X)\,V(Y)}} = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y} = \rho(Y,X)$

if $\sigma_X > 0$ and $\sigma_Y > 0$.

If $\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2\left[\begin{pmatrix} \mu_Y \\ \mu_X \end{pmatrix}, \begin{pmatrix} \sigma_Y^2 & \sigma_{YX} \\ \sigma_{XY} & \sigma_X^2 \end{pmatrix}\right]$,

then $\sigma_{YX} = \mathrm{cov}(Y,X) = \mathrm{cov}(X,Y) = \sigma_{XY}$.

ex] Know for E2 $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left[\begin{pmatrix} 1 \\ 17 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right]$

a) Which pairs of variables are ind?

$X_1 \perp\!\!\!\perp X_3$ and $X_2 \perp\!\!\!\perp X_3$

b) $\rho(X_1, X_2) = \dfrac{1}{\sqrt{4}\sqrt{3}} = .2887$

c) $\rho(X_1, X_3) = 0$

d) $\rho(X_2, X_3) = 0$

ex] $\underset{p\times 1}{\underset{\sim}{Y}} \sim N_p(\underset{\sim}{\mu}, \sigma^2 I)$. Find the dist of $\underset{q\times p}{A}\underset{\sim}{Y}$.

$p \times p$ identity matrix

Soln $A\underset{\sim}{Y} \sim N_q(\underset{\sim}{\mu}_A, \Sigma_A)$ where

$\underset{\sim}{\mu}_A = A\underset{\sim}{\mu}$ and $\Sigma_A = \sigma^2 AIA^T = \sigma^2 AA^T$.

p. 62

86] Know for E2 If $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$, then the conditional distribution

$$\underset{\sim}{X}_1 \mid \underset{\sim}{X}_2 = \underset{\sim}{x}_2 \sim N_q\left[\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{x}_2 - \underset{\sim}{\mu}_2), \underbrace{\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}_{\text{free of } X_2 = x_2}\right]$$

Notation] $$\underset{\sim}{X}_1 \mid \underset{\sim}{X}_2 \sim N_q\left[\underbrace{\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2)}_{\text{family of means depending on the value of } \underset{\sim}{X}_2}, \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right]$$

ex] Let $$\begin{pmatrix} Y \\ X_1 \\ \vdots \\ X_p \end{pmatrix} \sim N_{p+1}\left[\begin{pmatrix} E(Y) \\ E(\underset{\sim}{X}) \end{pmatrix}, \begin{pmatrix} \underset{1\times 1}{V(Y)} & \underset{1\times p}{\Sigma_{Y\underset{\sim}{X}}} \\ \Sigma_{\underset{\sim}{X}Y} & \underset{p\times p}{\Sigma_{\underset{\sim}{X}\underset{\sim}{X}}} \end{pmatrix}\right]$$

where $\Sigma_{\underset{\sim}{X}\underset{\sim}{X}} = \text{cov}(\underset{\sim}{X})$ and $\Sigma_{Y\underset{\sim}{X}} = \text{cov}(Y, \underset{\sim}{X})$

$= E\left[\underbrace{(Y - E(Y))}_{\text{scalar}}(\underset{\sim}{X} - E\underset{\sim}{X})^T\right]$.

Then $E(Y \mid \underset{\sim}{X} = \underset{\sim}{x}) = E(Y) + \Sigma_{Y\underset{\sim}{X}}\Sigma_{\underset{\sim}{X}\underset{\sim}{X}}^{-1}(\underset{\sim}{x} - E\underset{\sim}{X})$

and $V(Y \mid \underset{\sim}{X} = \underset{\sim}{x}) = V(Y) - \Sigma_{Y\underset{\sim}{X}}\Sigma_{\underset{\sim}{X}\underset{\sim}{X}}^{-1}\Sigma_{\underset{\sim}{X}Y}$ both are scalars

p. 63

87] Know bivariate normal $\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2\left[\begin{pmatrix} E(Y) \\ E(X) \end{pmatrix}, \begin{pmatrix} V(Y) & \text{cov}(Y,X) \\ \text{cov}(X,Y) & V(X) \end{pmatrix}\right]$

where $\text{cov}(X,Y) = \text{cov}(Y,X)$.

Then $Y|X=x \sim N_1\left[E(Y|X=x), V(Y|X=x)\right]$

where $E(Y|X=x) = E(Y) + \text{cov}(X,Y)\frac{1}{V(X)}(x - E(X))$

equation of a line in x

$= E(Y) + \rho(X,Y)\sqrt{\frac{V(Y)}{V(X)}}(x - E(X))$ and

$V(Y|X=x) = V(Y) - \text{cov}(X,Y)\frac{1}{V(X)}\text{cov}(X,Y)$

constant wrt x

$= V(Y) - \rho(X,Y)\sqrt{\frac{V(Y)}{V(X)}}\,\rho(X,Y)\sqrt{V(X)}\sqrt{V(Y)}$

$= V(Y) - [\rho(X,Y)]^2 V(Y) = V(Y)\left[1 - (\rho(X,Y))^2\right]$

ex] Know for E2 2.65 $\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2\left[\begin{pmatrix} 49 \\ 100 \end{pmatrix}, \begin{pmatrix} 16 & \sigma_{12} \\ \sigma_{12} & 25 \end{pmatrix}\right]$

a) If $\sigma_{12} = 0$ find $Y|X$.

b) If $\sigma_{12} = 10$, find $E(Y|X)$.

c) If $\sigma_{12} = 10$, find $V(Y|X)$.

soln] $\Sigma_{ij} = \sigma_{ij}$

a) $Y|X \sim N(49, 16)$ since $Y \perp\!\!\!\perp X$

b) $E(Y|X) = \mu_Y + \Sigma_{12}\Sigma_{22}^{-1}(X - \mu_X)$

$= 49 + \frac{10}{25}(X - 100) = 49 + \frac{2}{5}(X - 100) = \boxed{9 + 0.4X}$

c) $V(Y|X) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} =$

$16 - 10\frac{1}{25}10 = 16 - 4 = \boxed{12}$

Skip Section 2.10

Ch3 Exponential Families 18.5

1] p81-2 Know for final

A family of joint pmfs or pdfs $\{f(\underset{\sim}{y}|\underset{\sim}{\theta}) \mid \underset{\sim}{\theta} = \underbrace{(\theta_1, \ldots, \theta_j)}_{\text{unknown}} \in \Theta\}$ for a random vector $\underset{\sim}{Y}$ is an exponential

(*) family if $f(\underset{\sim}{y}|\underset{\sim}{\theta}) = h(\underset{\sim}{y})\, c(\underset{\sim}{\theta}) \exp\left[\sum_{i=1}^{k} w_i(\underset{\sim}{\theta})\, t_i(\underset{\sim}{y})\right]$

for all $\underset{\sim}{y}$ where $c(\underset{\sim}{\theta}) \geq 0$ and $h(\underset{\sim}{y}) \geq 0$. The functions c, h, t_i and w_i are real valued. Θ is the parameter space. It is crucial that $c, w_1, \ldots, w_K$ do not depend on $\underset{\sim}{y}$ and $h, t_1, \ldots, t_K$ do not depend on $\underset{\sim}{\theta}$.

could show $\forall \underset{\sim}{y} \in \mathcal{Y}$ too

The family is a K parameter exponential family if K is the smallest integer where (*) holds.

2] p82 know A RV Y is from an exponential family if $f(y|\underset{\sim}{\theta}) = h(y)\, c(\underset{\sim}{\theta}) \exp\left[\sum_{i=1}^{k} w_i(\underset{\sim}{\theta})\, t_i(y)\right]$.

3) p82 know A RV Y is from a 1 parameter exponential family if $f(y|\theta) = h(y)\, c(\theta) \exp[w(\theta)\, t(y)]$.

4) Can define $f(\underset{\sim}{y}|\underset{\sim}{\theta})$ $\forall\, \underset{\sim}{y}$ by taking $h(\underset{\sim}{y}) = g(\underset{\sim}{y})\, I_{\mathcal{Y}}(\underset{\sim}{y})$. Note that

$\mathcal{Y}$ can not depend on $\underset{\sim}{\theta}$ if $\underset{\sim}{Y}$ is from an exp fam.

Know

5] Suppose $Y_1, ..., Y_n$ are iid with support $\mathcal{Y}_i \equiv \mathcal{Y}^*$. Then $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n = \mathcal{Y}^* \times \cdots \times \mathcal{Y}^*$.

Then $I_{\mathcal{Y}}(\underline{y}) = I_{\mathcal{Y}_1}(y_1) \cdots I_{\mathcal{Y}_n}(y_n) = I[\text{all } y_i \in \mathcal{Y}^*]$

If $\mathcal{Y}_i$ does not depend on $\underline{\theta}$, then $I(\underline{y} \in \mathcal{Y}) = \prod_{i=1}^n I_{\mathcal{Y}^*}(y_i)$ is part of $h(\underline{y})$.

If $\mathcal{Y}_i$ does depend on $\underline{\theta}$, then the family is not an exp fam.

Know

6) Let $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$ be the order statistics. So $Y_{(1)} = \min(Y_1, ..., Y_n)$ and $Y_{(n)} = \max(Y_1, ..., Y_n)$. If $\mathcal{Y}^* = (\theta_1, \theta_2]$, and $Y_1, ..., Y_n$ are iid, then

$I_{\mathcal{Y}}(\underset{\sim}{y}) = I[\text{all } y_i \in (\theta_1, \theta_2]] =$

$I[\theta_1 < y_{(1)} \le y_{(n)} \le \theta_2] = I[\theta_1 < y_{(1)}]\, I[y_{(n)} \le \theta_2]$.

$\mathcal{Y}^* = (a, b)$, $a = -\infty, b = \infty$ possible, is similar.

In general $I_A(\underline{y})\, I_B(\underline{y}) = I_{A \cap B}(\underline{y})$, and $\prod_{i=1}^n I_{A_i}(\underline{y}) = I_{\bigcap_{i=1}^n A_i}(\underline{y})$.

7] Know p. 82 To demonstate that $f(\underset{\sim}{y} | \underline{\theta})$ is an exp fam, find $h(\underline{y})$, $c(\underline{\theta})$, $w_i(\underline{\theta})$, and $t_i(\underline{y})$ so that 1] or 2] or 3] holds.

ex (special case of Th 3.1)

Suppose $Y_1, ..., Y_n$ are iid from an

19.5

exp fam $f(y|\underset{\sim}{\theta}) = h(y)\, c(\underset{\sim}{\theta}) \exp\left[\sum_{i=1}^{k} w_i(\underset{\sim}{\theta})\, t_i(y)\right]$,

Then the joint dist of $Y_1, \dots, Y_n$ follows an exp family.

proof $f(y_1, \dots, y_n) = \prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n} h(y_i)\, c(\underset{\sim}{\theta}) \exp\left[\sum_{j=1}^{k} w_j(\underset{\sim}{\theta})\, t_j(y_i)\right]$

$$= \left[\prod_{i=1}^{n} h(y_i)\right] \left[c(\underset{\sim}{\theta})\right]^n \exp\left(\sum_{i=1}^{n} \left[\sum_{j=1}^{k} w_j(\underset{\sim}{\theta})\, t_j(y_i)\right]\right)$$

$$= \underbrace{\left[\prod_{i=1}^{n} h(y_i)\right]}_{h^*(\underset{\sim}{y})} \underbrace{\left[c(\underset{\sim}{\theta})\right]^n}_{c^*(\underset{\sim}{\theta})} \exp\left[\sum_{j=1}^{k} \underbrace{w_j(\underset{\sim}{\theta})}_{w_j^*(\theta)} \underbrace{\left(\sum_{i=1}^{n} t_j(y_i)\right)}_{t_j^*(\underset{\sim}{y})}\right]$$

$Y \sim MB(\mu, \sigma)$ ex} Suppose $f(y) = \dfrac{\sqrt{2}\,(y-\mu)^2}{\sigma^3\sqrt{\pi}}\, e^{-\frac{1}{2\sigma^2}(y-\mu)^2}\, I(y \geq \mu)$.

$\Theta = \{(\mu, \sigma) \mid \mu \in \mathbb{R}, \sigma > 0\}$

a) If μ is unknown, this is not an exp fam since $\mathcal{Y} = (\mu, \infty)$ depends on μ.

b) If μ is known, $\theta = \sigma$, family is $\{f(y|\sigma) \mid \sigma > 0\}$ and

$$f(y) = \underbrace{\frac{\sqrt{2}}{\sqrt{\pi}}(y-\mu)^2 I(y \geq \mu)}_{h(y) \geq 0} \underbrace{\frac{1}{\sigma^3}}_{c(\sigma) > 0} \exp\left[\underbrace{\frac{-1}{2\sigma^2}}_{w(\sigma)} \underbrace{(y-\mu)^2}_{t(y)}\right].$$

So this is a 1 parameter exp family for each value of μ.

8] p 81 Typically an exp fam has 3 parameterizations

i) The usual parameterization, with parameter space Θ, is given in ch 10.