ii) The parameterization for the k parameter exp family given in 1], 2] or 3].

iii) The natural parameterization given below is used for theory.

9) Know for final p 83 Let Ω be the natural parameter space for $\underset{\sim}{\eta}$. The natural parameterization for an exp fam is

$$f(\underline{y} \mid \underline{\eta}) = h(\underline{y})\, b(\underset{\sim}{\eta}) \exp\left[\sum_{i=1}^{k} \eta_i\, t_i(\underline{y})\right]$$

where $\underset{\sim}{\eta} \in \Omega$ and $h(\underline{y})$ and $t_i(\underline{y})$

are the same as in 1].

10] Know For RV Y the natural parameterization is $f(y/\underline{\eta}) = h(y)\; b(\underline{\eta}) \exp\left[\sum_{i=1}^{k} \eta_i\, t_i(y)\right]$

and if $k=1$, $f(y/\eta) = h(y)\; b(\eta) \exp[\eta\; t(y)]$.

11] p 83 Let $\tilde{\Omega} = \left\{ \underline{\eta} \,\middle|\, \frac{1}{b(\underline{\eta})} = \int_{-\infty}^{\infty} \underbrace{h(y) \exp\left(\sum_{i=1}^{k} \eta_i\, t_i(y)\right)}_{\text{kernel function}} dy < \infty \right\}$

Replace the integral by a sum for a pmf.

12] Know p 83 Suppose E1: $\Omega = \tilde{\Omega}$

E2: in the natural parameterization, neither the η_i nor the t_i satisfy a linearity constraint.

E3: Ω is a k dimensional open set, If E1), E2), and E3) hold, then the family is a k parameter regular exponential family kP-REF.

Note] If E1 and E2 hold, then $f(y)$ is a k par full exp fam.

13] p 83 Let $d_i(x)$ denote $t_i(y)$, $w_i(\theta)$ or η_i. A linearity constraint is satisfied by $d_1(x), \ldots, d_k(x)$ if $\sum_{i=1}^{k} a_i d_i(x) = c$ for some constants a_i and c and for all x in the support = samplespace or parameter space where not all $a_i = 0$. If $\sum_{i=1}^{k} a_i d_i(x) = c \ \forall x$ only if $a_1 = \cdots = a_k = 0 = c$, then the $d_i(x)$ do not satisfy a linearity constraint.

Tip: for a 1parameter family, only constant functions satisfy the linearity constraint.

14] Random vectors can belong to a kP-REF: replace y by $\underset{\sim}{y}$ everywhere. There are many nice theorems for REFs.

15] know p 85-6 For exp families in Ch 10, other than the χ^2_p and inverse Gaussian distributions, suppose $\dim \Theta = k = \dim(\Omega)$. Assume $\eta_i = w_i(\underset{\sim}{\theta})$. Assume the usual parameter space Θ_U is as big as possible ($\Theta_U = \{ \underset{\sim}{\theta} \in \mathbb{R}^k \mid \int f(y|\underset{\sim}{\theta}) dy = 1 \}$)

and let $\Theta = \{\underline{\theta} \in \Theta_U \mid w_1(\underline{\theta}), \ldots, w_k(\underline{\theta})$ are defined$\}$. Then assume the natural parameter space $\Omega = \tilde{\Omega}$ (E1) and

$\Omega = \{(\eta_1, \ldots, \eta_k) \mid \eta_i = w_i(\underline{\theta})$ for $\underline{\theta} \in \Theta\}$.

Tip] $f(y)$ should have the same formula $\forall\, \underline{\theta} \in \Theta$ eg $f(y) = \rho^y(1-\rho)^{1-y}$ $y \in \{0,1\} \Rightarrow \Theta = (0,1)$

$\rho = P(Y=1)$
$\rho = 0 \Rightarrow$
$P(Y=0) = 1$
$\rho = 1 \Rightarrow$
$P(Y=1) = 1$

16] **Know** To show a dist $\rho^y(1-\rho)^{1-y}$ is a kP-REF, show that the family is a k par exp fam, set $\eta_i = w_i(\underline{\theta})$ and find the range of R_i of η_i. Then $\Omega = R_1 \times \cdots \times R_k$.
Usually $k=1$ and Ω is an open interval, or $k=2$ and Ω is a cross product of 2 open intervals.

Tips $a = e^{\log(a)}$ for $a > 0$ and $t_i(y) \in (0,\infty)$ is better than $t_i(y) \in (-\infty, 0)$.

ex] see ex 3.1 – 3.5 and p 84-5 – ch 10 dist's

ex] $Y \sim EXP(\lambda)$, $f(y) = \frac{1}{\lambda} e^{-y/\lambda}$, $y > 0$, $\lambda > 0$

$$= \underbrace{\frac{1}{\lambda}}_{c(\lambda)} \underbrace{I[y>0]}_{h(y)} \exp\Big[\underbrace{\frac{-1}{\lambda}}_{w(\lambda)} \underbrace{y}_{t(y)}\Big] = -\eta \underbrace{I(y>0)}_{\text{usually omitted}} \exp[\eta y]$$

wherever λ or $w(\lambda)$ appears replace $w(\lambda)$ by η and λ by $w^{-1}(\eta)$: $\eta = \frac{-1}{\lambda}$ so $\frac{1}{\lambda} = -\eta$ or $\lambda = \frac{-1}{\eta}$

so $\eta = -\frac{1}{\lambda}$ for $\eta \in \Omega = (-\infty, 0)$

and $f(y)$ is a 1P-REF.

ex] $Y \sim chi(p, \sigma)$, $f(y) = \dfrac{y^{p-1} e^{\frac{-1}{2\sigma^2} y^2}}{\sigma^p \, 2^{\frac{p}{2}-1} \, \Gamma(\frac{p}{2})}$, $y > 0$, $\sigma, p > 0$.

So $\Theta_U = \{(\sigma, p) \mid \sigma > 0, p > 0\} = (0,\infty) \times (0,\infty)$.

Now $f(y) = \underbrace{\frac{1}{2^{\frac{p}{2}-1}\Gamma(\frac{p}{2})\sigma^p}}_{c(p,\sigma)} \underbrace{I(y>0)}_{h(y)} \exp\left[\underbrace{(p-1)}_{w_1(p,\sigma)}\underbrace{\log(y)}_{t_1(y)} \underbrace{-\frac{1}{2\sigma^2}}_{w_2(p,\sigma)}\underbrace{y^2}_{t_2 y}\right]$

So $\eta_1 = p-1$, $\eta_2 = \frac{-1}{2\sigma^2}$ and $\Omega = (-1,\infty) \times (-\infty,0)$

and $f(y)$ is a 2P-REF.

17] The exp fam parameterizations are not unique.
For $a \neq 0$ $\eta_i t_i(y) = (a\eta_i)\frac{t_i(y)}{a} = \tilde{\eta}_i \tilde{t}_i(y)$,

* Would like $t_i(y)$ to have a brand name dist.

18) i) cauchy, F, logistic, t, uniform dist's can't be put in exp fam form.
ii) if support depends on unknown $\underset{\sim}{\theta}$, family is not an exp fam.
iii) p 88–9 watch out for k par exp families where $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$ is completely determined by $d < k$ of its elements.
These families are not regular because the "natural parameter space" Ω_c places a restriction on Ω such that Ω_c is not a k dim open set. Here $\dim(\Theta) = d < \dim(\Omega) = k$.

19] A k dim open set contains a k dim rectangle. For $k = 1$, a rectangle is an interval.

ex] ex 3.8, p. 89 $Y \sim N(\theta, \theta^2)$ $d=1<k=2$, $\theta \neq 0$

$f(y) = \frac{1}{\sqrt{2\pi}\,\theta} \exp\left(\frac{-(y-\theta)^2}{2\theta^2}\right) =$

$\frac{1}{\sqrt{2\pi}} \frac{1}{\theta} \exp\left[\frac{-1}{2\theta^2}(y^2 - 2\theta y + \theta^2)\right]$

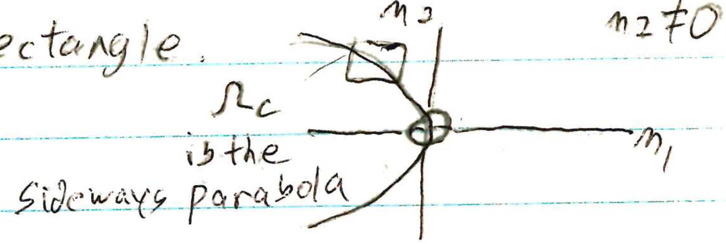
$$= \underbrace{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}}}_{h(y)} \underbrace{\frac{1}{\theta}}_{c(\theta)} \exp\Big[\underbrace{\frac{-1}{2\theta^2}}_{\eta_1 = \frac{-1}{2\theta^2}} y^2 + \underbrace{\frac{1}{\theta}}_{\eta_2 = \frac{1}{\theta}} y\Big]$$ and the

So $\eta_1 = -\frac{1}{2}\eta_2^2$

"natural parameterspace" $\Omega_c = \{(\eta_1, \eta_2) \mid \eta_1 = -\frac{1}{2}\eta_2^2,\ -\infty < \eta_1 < 0,\ \eta_2 \in \mathbb{R}/\{0\}\}$

$\eta_2 \neq 0$

is not a 2 dim open set and does not contain a 2 dim open rectangle.



§3.1 20] Let Y have natural parameterization

$$f(y \mid \underline{\eta}) = h(y)\, b(\underline{\eta}) \exp\Big[\sum_{i=1}^{k} \eta_i t_i(y)\Big]$$

$$= h(y) \exp\Big[\sum_{i=1}^{k} \eta_i t_i(y) - a(\underline{\eta})\Big]$$

Where $a(\underline{\eta}) = -\log(b(\underline{\eta}))$.

21] p 89-90 Derivative operators of all orders and integral or sum operators can be interchanged if the integrand or summand is $g(\underline{y}) f(\underline{y} \mid \underline{\eta})$ or $g(\underline{y}) f(\underline{y} \mid \underline{\theta})$ and

$E|g(\underline{Y})|$ exists. So for $j \geq 1$,

$$\frac{\partial^j}{\partial \eta_i^j} \sum_k g(y_k) \overbrace{f(y_k \mid \underline{\eta})}^{\text{exp fam}} = \sum g(y_k) \frac{\partial^j}{\partial \eta_i^j} f(y_k \mid \underline{\eta}).$$

Similarly $\frac{\partial}{\partial \ldots} \int g(y) \overbrace{f(y \mid \theta)}^{\text{exp fam}} dy = \int g(y) \frac{\partial}{\ldots} f(y \mid \theta) dy$

22] If Y comes from exp fam, 20],

then $\underset{\sim}{T} = (t_1(Y), \ldots, t_k(Y))$ is a well behaved random vector. In particular

$$E[t_i(Y)] = \frac{\partial}{\partial \eta_i} a(\underset{\sim}{\eta}) = -\frac{\partial}{\partial \eta_i} \log(b(\underset{\sim}{\eta})),$$

$$\text{cov}(t_i(Y), t_j(Y)) = \frac{\partial^2}{\partial \eta_i \partial \eta_j} a(\underset{\sim}{\eta}) = -\frac{\partial^2}{\partial \eta_i \partial \eta_j} \log(b(\underset{\sim}{\eta})),$$

$V(t_i(Y)) = \text{cov}(t_i(Y), t_i(Y))$ and if $\underset{\sim}{\eta} \in$ interior Ω, then the mgf of $\underset{\sim}{T}$

true if Ω is an open set

is $m_{\underset{\sim}{T}}(\underset{\sim}{s}) = \exp[a(\underset{\sim}{\eta} + \underset{\sim}{s}) - a(\underset{\sim}{\eta})]$

for $\underset{\sim}{s}$ in some neighborhood of $\underset{\sim}{0}$.

Skim proofs of Th 3.3, 3.4.

23] If $Y_1, \ldots, Y_n$ are iid from a 1P-REF, $T \equiv T_n = \sum_{i=1}^{n} t(Y_i)$ has a 1 parameter exp fam.

Note] Th 3.6 & 3.7 give the dist of $\sum_{i=1}^{n} t(Y_i)$ for many 1P-REFs.

24] p. 92 If $Y_1, \ldots, Y_n$ are iid from a kP-REF with $f(y|\underset{\sim}{\eta}) = h(y)\, b(\underset{\sim}{\eta}) \exp[\sum_{i=1}^{k} \eta_i t_i(Y)]$, and $T_i(Y_1, \ldots, Y_n) = \sum_{j=1}^{n} t_i(Y_j)$ for $i = 1, \ldots, k$, then

$\underset{\sim}{T} = (T_1, \ldots, T_k)$ has a k par exp fam with

$$f(\underset{\sim}{t}|\underset{\sim}{\eta}) = h^*(\underset{\sim}{t})\,[b(\underset{\sim}{\eta})]^n \exp\left[\sum_{i=1}^{K} \eta_i t_i\right].$$

Ch4

1] * p 101 A Statistic is a function of the data that does not depend on any unknown parameters. The prob dist of the statistic is called the sampling distribution of the statistic.

2] $Y_1, \ldots, Y_n$ = data

$y_1, \ldots, y_n$ = (observed) data

$\underset{\sim}{T}(\underset{\sim}{Y}) = (T_1(\underset{\sim}{Y}), \ldots, T_K(\underset{\sim}{Y}))$ = statistic = random vector $\underset{\sim}{RV}$

$\underset{\sim}{T}(\underset{\sim}{y}) = (T_1(\underset{\sim}{y}), \ldots, T_K(\underset{\sim}{y}))$ = (observed) statistic

Note: The domain of $\underset{\sim}{T}$ includes the support $\mathcal{Y}$.

3) know p 102 sample mean $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$, $n \geq 2$

sample standard deviation $S = \sqrt{S^2}$

$Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$ are the order statistics

4) know (p 53) If $Y_1, \ldots, Y_n$ are iid with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$, then $E(\bar{Y}) = \mu$, $V(\bar{Y}) = \sigma^2/n$ and $E(S^2) = \sigma^2$.

proof $E\bar{Y} = \frac{1}{n}\sum E Y_i = \frac{1}{n} n\mu = \mu$

$V\bar{Y} = V\left(\frac{1}{n}\sum Y_i\right) = \frac{1}{n^2} V\left(\sum Y_i\right) = \frac{1}{n^2}\sum V(Y_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$

Now $\Sigma(y_i-\bar{y})^2 = \Sigma y_i^2 - n(\bar{y})^2$. (*)

So $E(S^2) = \frac{1}{n-1} E\left[\Sigma(Y_i-\bar{Y})^2\right] = \frac{1}{n-1} E\left[\Sigma Y_i^2 - n(\bar{Y})^2\right]$

$= \frac{1}{n-1}\left[n E(Y_1^2) - n E[(\bar{Y})^2]\right]$

$= \frac{1}{n-1}\left[n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right] =$

$\frac{1}{n-1}(n\sigma^2 - \sigma^2) = \sigma^2$.

5] Know Let $Y_1, \ldots, Y_n$ be iid with mgf $m_Y(t)$ for $|t| < t_0$. Then

$$m_{\bar{Y}}(t) = \left[m_Y\left(\frac{t}{n}\right)\right]^n.$$

proof $m_{\bar{Y}}(t) = E\left[e^{t(Y_1+\cdots+Y_n)/n}\right] = E\left[e^{tY_1/n} \cdots e^{tY_n/n}\right]$

$= \prod_{i=1}^n E\left(e^{\frac{t}{n}Y_i}\right) = \prod_{i=1}^n m_Y\left(\frac{t}{n}\right) = \left[m_Y\left(\frac{t}{n}\right)\right]^n$

for $\left|\frac{t}{n}\right| < t_0$ or $|t| < t_0 n$.

6] Know for final Th 4.1 p. 122 Let $Y_1, \ldots, Y_n$ be iid $N(\mu, \sigma^2)$.

a) $\bar{Y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

b) $\bar{Y} \perp\!\!\!\perp S^2$

c) $(n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{n-1}$ so $\sum_{i=1}^n (Y_i-\bar{Y})^2 \sim \sigma^2\chi^2_{n-1}$

proof See text. Also get a) with mgfs and b) $(Y_1, \ldots, Y_n)^T \sim N_n(\underset{\sim}{\mu}, \sigma^2 I_n)$

So $\begin{pmatrix} \overline{Y} \\ Y_1 - \overline{Y} \\ \vdots \\ Y_n - \overline{Y} \end{pmatrix}$ is also MVN

ie $A = -\begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & & \frac{1}{n} \end{bmatrix} + \begin{bmatrix} \frac{2}{n} & \cdots & \frac{2}{n} \\ 1 & 0 \cdots & 0 \\ 0 & 1 \cdots & 0 \\ 0 & \cdots 0 & 1 \end{bmatrix}$

linear combo of MVN is MVN

Now $\text{cov}(\overline{Y}, Y_k - \overline{Y}) = \text{cov}\left(\frac{1}{n}\sum_{i=1}^n Y_i,\ Y_k - \frac{1}{n}\sum_{j=1}^n Y_j\right)$

$= \frac{1}{n}\sum_{i=1}^n \text{cov}(Y_i, Y_k) - \frac{1}{n^2}\sum_i \sum_j \text{cov}(Y_i, Y_j)$

$= \frac{1}{n} V(Y_k) - \frac{1}{n^2}\sum_{i=1}^n V(Y_i) = \frac{\sigma^2}{n} - \frac{n\sigma^2}{n^2} = 0.$

$\text{cov}(Y_i, Y_j) = 0,\ i \neq j$

Hence $\overline{Y} \perp\!\!\!\perp Y_i - \overline{Y}$ for $i = 1, \ldots, n$.

$S^2 = \frac{1}{n-1}\sum (Y_i - \overline{Y})^2$ is a function of $Y_i - \overline{Y}$.

So $S^2 \perp\!\!\!\perp \overline{Y}$.

7) know Th 4.2 Let $Y_1, \ldots, Y_n$ be iid with cdf F_Y and pdf f_Y. Then a) the pdf of the max is $f_{Y_{(n)}}(t) = n[F_Y(t)]^{n-1} f_Y(t)$ and b) the pdf of the min is $f_{Y_{(1)}}(t) = n[1 - F_Y(t)]^{n-1} f_Y(t)$.

proof a) $F_{Y_{(n)}}(t) = P(Y_{(n)} \leq t) = P(Y_1 \leq t, \ldots, Y_n \leq t)$

$= \prod_{i=1}^n P[Y_i \leq t] = [F_Y(t)]^n$. So $f_{Y_{(n)}}(t) = \frac{d}{dt} F_{Y_{(n)}}(t)$

$= n[F_Y(t)]^{n-1} f_Y(t)$.

b) $F_{Y_{(1)}}(t) = P(Y_{(1)} \leq t) = 1 - P(Y_{(1)} > t) = 1 - P[Y_1 > t, \ldots, Y_n > t]$

$= 1 - \prod P(Y_i > t) = 1 - [1 - F_Y(t)]^n$. So $f_{Y_{(1)}}(t) = \frac{d}{dt} F_{Y_{(1)}}(t)$

$= -n[1 - F_Y(t)]^{n-1}[-f_Y(t)]$

8] <u>E2 problem</u> Find $F_{Y_{(1)}}(t)$ or $f_{Y_{(n)}}(t)$ 24.5

and then find $E[Y_{(1)}]$ or $E[Y_{(n)}]$.

ex) see ex 4.4

ex] If $Y_{(1)}, \ldots, Y_n$ are iid $EXP(1)$, find $E[Y_{(1)}]$.

soln $F_Y(y) = 1 - e^{-y}$, $y > 0$ so

$P(Y_{(1)} \le t) = 1 - P(Y_{(1)} > t) = 1 - [P(Y_1 > t)]^n$

$= 1 - [1 - F_Y(t)]^n = 1 - [1 - (1 - e^{-t})]^n = 1 - e^{-nt}$

$= 1 - e^{-t/\frac{1}{n}}$, $t > 0$ $\Rightarrow Y_{(1)} \sim EXP(\frac{1}{n})$

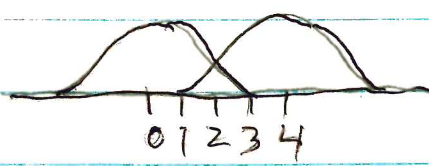
so $E[Y_{(1)}] = \frac{1}{n}$.

better — Also, $f_{Y_{(1)}}(t) = n\,(1 - F_Y(t))^{n-1} f_Y(t) = n\,[1 - (1 - e^{-t})]^{n-1} e^{-t}$

$= n\,[e^{-t}]^{n-1} e^{-t} = \underbrace{n e^{-nt}}_{EXP(\frac{1}{n})\text{ pdf}}$, $t > 0$.

9] Model $\{f(y|\underset{\sim}{\theta}) \mid \underset{\sim}{\theta} \in \Theta\}$ where $\underset{\sim}{\theta}$ is unknown. The data $Y_1, \ldots, Y_n$ is used to gain information about $\underset{\sim}{\theta}$. Sometimes use $f_{\underset{\sim}{\theta}}(y) = f(y|\underset{\sim}{\theta})$

ex] $\theta \in \Theta = \{0, 4\}$, $f(y|\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\theta)^2} \sim N(\theta, 1)$



If observed $y < 2$, $\theta = 0$ is more reasonable than $\theta = 4$.

10] Want to reduce the data $Y_1, \ldots, Y_n$ to just a few statistics $T_1(\underset{\sim}{Y}), \ldots, T_k(\underset{\sim}{Y})$ (where $1 \le k << n$) without losing any information about $\underset{\sim}{\theta}$.

11] p 107 The basic idea of a sufficient statistic $\underline{T}(\underline{Y})$ for $\underline{\theta}$ is that all of the information needed for inference from the data $Y_1, \ldots, Y_n$ about $\underline{\theta}$ is contained in $\underline{T}(\underline{Y})$.

12] know p108 Let $\{f(\underline{y}|\underline{\theta}) \mid \underline{\theta} \in \Theta\}$ be a family of distributions. A statistic $\underline{T}(\underline{Y})$ is a sufficient statistic for $\underline{\theta}$ if

the conditional distribution of $\underline{Y}$ given $\underline{T}(\underline{Y}) = \underline{t}$ does not depend on $\underline{\theta}$ for any $\underline{t}$ in the support of $\underline{T}$.

Note $f(\underline{y} \mid T(\underline{y}), \underline{\theta}) \equiv f(\underline{y} \mid T(\underline{y}))$ is free of $\underline{\theta}$ $\forall$ $\underline{\theta} \in \Theta$.

(or $f_{\underline{\theta}}(\underline{y} \mid T(\underline{y})) \equiv f(\underline{y} \mid T(\underline{y}))$, RHS free of $\underline{\theta}$)

Note Often $Y_1, \ldots, Y_n$ are iid or at least independent.

Note] $\underline{T}(\underline{Y})$ and $\underline{\theta}$ can be scalars, vectors or matrices. eg $\underline{\theta} = (\underline{\mu}\ \Sigma)$ for $\underline{Y}$ ~ MVN.

13] know for final p108 The way to demonstrate that $\underline{T}(\underline{Y})$ is a suff stat for $\underline{\theta}$ is to use the Factorization Theorem:
Let $\{f(\underline{y}|\underline{\theta}) \mid \underline{\theta} \in \Theta\}$ be a family of pdfs or pmfs. A statistic $\underline{T}(\underline{Y})$ is a sufficient statistic for $\underline{\theta}$ iff $\forall$ sample points $\underline{y}$ and $\forall$ $\underline{\theta} \in \Theta$,

$$f(\underline{y} \mid \underline{\theta}) = g(\underline{T}(\underline{y}) \mid \underline{\theta})\ h(\underline{y})$$

where $g \geq 0$, $h \geq 0$, h does not depend on $\underline{\theta}$ and

g depends on $\underline{y}$ only through $\underset{\sim}{T}(\underline{y})$. 25.5

14] When asked to find a suff stat for $\underline{\theta}$ when $Y_1, \ldots, Y_n$ are iid with pdf or pmf $f(y|\underline{\theta})$, find $f(\underline{y}|\underline{\theta}) = \prod_{i=1}^{n} f(y_i|\underline{\theta})$, then apply the Factorization th.

Know for E2

ex] 4.7 $Y_1, \ldots, Y_n$ iid $N(\mu, \sigma^2)$

$$f(\underline{y}|\underline{\theta}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{y_i-\mu}{\sigma}\right)^2} = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{\frac{-1}{2\sigma^2}\sum(y_i-\mu)^2}$$

$$= \underbrace{\frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\frac{-1}{2\sigma^2}\left(\sum y_i^2 - 2\mu\sum y_i + n\mu^2\right)}_{g(\underset{\sim}{T}(\underline{y})|\underline{\theta})} \cdot \underbrace{1}_{h(\underline{y})}$$

Thus $\underset{\sim}{T}(\underline{Y}) = \left(\sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} Y_i^2\right)$ is a sufficient statistic for $\underline{\theta} = (\mu, \sigma^2)$ by factorization.

15] Know p110: If $Y_1, \ldots, Y_n$ are iid with $f(y|\underline{\theta}) = k(y|\underline{\theta})\, I(y \in \mathcal{Y}^*)$ then $f(\underline{y}|\underline{\theta}) = \prod_{i=1}^{n} f(y_i|\underline{\theta}) = \prod_{i=1}^{n} k(y_i|\underline{\theta}) \prod_{i=1}^{n} I(y_i \in \mathcal{Y}^*)$.

Now $\prod_{i=1}^{n} I(y_i \in \mathcal{Y}^*) = I(\underline{y} \in \mathcal{Y}) = I(\text{all } y_i \in \mathcal{Y}^*)$

where $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n = \mathcal{Y}^* \times \cdots \times \mathcal{Y}^*$.

If $\mathcal{Y}^*$ does not depend on $\underline{\theta}$, then $\prod I(y_i \in \mathcal{Y}^*)$ is part of $h(\underline{y})$. If $\mathcal{Y}^*$ depends on unknown $\underline{\theta}$,