a) $(Y_{(1)}, Y_{(n)})$ is complete.

b) $Y_{(1)}$ is complete if θ_2 is known

c) $Y_{(n)}$ is complete if θ_1 is known.

iv) If $\underset{1 \times k}{\underset{\sim}{T}(\underset{\sim}{Y})}$ is not minimal suff, then

$\underset{\sim}{T}(\underset{\sim}{Y})$ is not complete.

30] Know rule of thumb p137: (can't use for a proof, but generally true) If $\underset{\sim}{T}(\underset{\sim}{Y})$ is $1 \times k$ and $\underset{\sim}{\theta}$ is $1 \times j$ where $k > j$, generally $\underset{\sim}{T}$ is not complete, especially if $k = 2$ and $j = 1$.
Watch out for $N(\theta, \theta^2)$, $N(\sigma, \sigma^2)$, $U(a(\theta), b(\theta))$ data.

31] Showing $\underset{\sim}{T}(\underset{\sim}{Y})$ is complete when $\underset{\sim}{Y}$ is not from a k dim exp fam where Ω contains a k dim rectangle (eg a KPREF), is generally very hard. You need to find every g such that $E_{\underset{\sim}{\theta}}[g(\underset{\sim}{T})] = 0 \ \forall \underset{\sim}{\theta} \in \Theta$

32] Showing $\underset{\sim}{T}(\underset{\sim}{Y})$ is not complete is usually very hard. You need to show that $\underset{\sim}{T}$ is not minimal sufficient or find a g where $E_\theta[g(\underset{\sim}{T}(\underset{\sim}{Y}))] = 0 \ \forall \underset{\sim}{\theta} \in \Theta$

but $P[g(\underset{\sim}{T}(\underset{\sim}{Y})) \neq 0] > 0$ or $P(g(\underset{\sim}{T}(\underset{\sim}{Y})) = 0) < 1$.
Use point 30] to get some partial credit.

33] Know for final: often if $\underset{\sim}{T}$ is not complete, there is some restriction that gives a clue to g.

$T = (\bar{Y}, S^2)$

ex] $Y \sim N(\mu, \mu^2)$ or $N(\sigma^2, \sigma^2)$, try $\bar{Y} - S^2$.

ex] $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left[\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}\right]$, $\theta = (\mu, \sigma_x^2, \sigma_y^2)$

$$f(x,y) = \frac{1}{2\pi \sigma_x \sigma_y} \exp\left\{ \frac{-1}{2}\left[\left(\frac{x-\mu}{\sigma_x}\right)^2 + \left(\frac{y-\mu}{\sigma_y}\right)^2\right]\right\} =$$

$$\frac{1}{2\pi\sigma_x\sigma_y} \exp\left[\frac{-1}{2}\left(\frac{\mu^2}{\sigma_x^2} + \frac{\mu^2}{\sigma_y^2}\right)\right] \exp\left[\frac{\mu}{\sigma_x^2} x + \frac{\mu}{\sigma_y^2} y - \frac{1}{2\sigma_x^2} x^2 - \frac{1}{2\sigma_y^2} y^2\right]$$

is a 4 parameter exp fam. Note $\dim(\underset{\sim}{T}) = 4 > 3 = \dim(\underset{\sim}{\theta})$.

$$\underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{y}) = \left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i, \sum_{i=1}^n x_i^2, \sum_{i=1}^n y_i^2\right)$$

$\underset{\sim}{T}$ is min suff i) because the η_i do not satisfy a linearity constraint

ii) By LSM $R[(\underset{\sim}{x}, \underset{\sim}{y}), (\underset{\sim}{w}, \underset{\sim}{z})] = f(\underset{\sim}{x}, \underset{\sim}{y}) / f(\underset{\sim}{w}, \underset{\sim}{z}) =$

$$\exp\left[\frac{\mu}{\sigma_x^2}(\Sigma x_i - \Sigma w_i)\right] \exp\left[\frac{\mu}{\sigma_y^2}(\Sigma y_i - \Sigma z_i)\right] \exp\left[\frac{-1}{2\sigma_x^2}(\Sigma x_i^2 - \Sigma w_i^2)\right] \exp\left[\frac{-1}{2\sigma_y^2}(\Sigma y_i^2 - \Sigma z_i^2)\right]$$

is constant $\forall \underset{\sim}{\theta}$ iff $\underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{y}) = \underset{\sim}{T}(\underset{\sim}{w}, \underset{\sim}{z})$ since $e^a > 0$.

The clue that $\underset{\sim}{T}$ is not complete is $\mu_x = \mu_y \equiv \mu$.

$$E_{\underset{\sim}{\theta}}[g(\underset{\sim}{T})] = E_{\underset{\sim}{\theta}}(\Sigma X_i - \Sigma Y_i) = n\mu - n\mu = 0 \quad \forall \underset{\sim}{\theta}.$$

So $g(\underset{\sim}{T}) = \Sigma X_i - \Sigma Y_i$, but $P_{\underset{\sim}{\theta}}(\Sigma X_i - \Sigma Y_i = 0) = P_{\underset{\sim}{\theta}}(g(\underset{\sim}{T}) = 0) = 0 < 1$, so $\underset{\sim}{T}$ is not complete.

$|\Sigma X$)

34] Know for final. A common problem is $X_1, \dots, X_n$ are iid $U(a(\theta), b(\theta))$. Find a min suff stat $\underset{\sim}{T}$. Determine whether $\underset{\sim}{T}$ is complete. Typically $\underset{\sim}{T}_2 = (X_{(1)}, X_{(n)})$ is 2 dimensional but θ 1 dimensional and $\underset{\sim}{T}_2$ is not complete. Here is an exception.

ex] ex. 4.16 $X_1, \dots, X_n$ are iid $U(1-\theta, 1+\theta)$, $\theta > 0$.

So $f_\theta(x) = \dfrac{1}{2\theta}$, $1-\theta < x < 1+\theta$.

$I(1-\theta < X_{(1)} < X_{(n)} < 1+\theta)$

$T_1 = \max(X_{(n)} - 1, 1 - X_{(1)})$ is min suff

show T_2 is not min suff.

$$f_{X_{(n)}}(t) = \frac{n}{2\theta}\left(\frac{t+\theta-1}{2\theta}\right)^{n-1}, \quad 1-\theta < t < 1+\theta$$

and

$$E_\theta[X_{(n)}] = \int x f_{X_{(n)}}(x)\,dx = \int_{1-\theta}^{1+\theta} x \frac{n}{2\theta}\left(\frac{x+\theta-1}{2\theta}\right)^{n-1} dx.$$

Let $u = \dfrac{x+\theta-1}{2\theta}$, $x = 2\theta u + 1 - \theta$. So $x = 1+\theta \to u = 1$ and $x = 1-\theta \to u = 0$, $dx = 2\theta\, du$.

$$\text{So } E_\theta(X_{(n)}) \overset{I}{=} n \int_0^1 \frac{2\theta u + 1 - \theta}{2\theta} u^{n-1}\, 2\theta\, du$$

$$= 2\theta n \int_0^1 u^n du + n(1-\theta)\int_0^1 u^{n-1} du$$

$$= 2\theta n \frac{u^{n+1}}{n+1}\Big|_0^1 + n(1-\theta)\frac{u^n}{n}\Big|_0^1 =$$

$$2\theta \frac{n}{n+1} + \frac{n(1-\theta)}{n} = 1-\theta + 2\theta\frac{n}{n+1} \approx 1+\theta$$

as you should expect

So $f_{X_{(1)}}(t) = -n \left(1 - \frac{t+\theta-1}{2\theta}\right)^{n-1} \left(-\frac{1}{2\theta}\right) = \frac{n}{2\theta}\left(\frac{\theta - t + 1}{2\theta}\right)^{n-1}$

for $1-\theta < t < 1+\theta$. So

$$E_\theta[X_{(1)}] = \int x f_{X_{(1)}}(x)\,dx = \int_{1-\theta}^{1+\theta} \frac{n}{2\theta}\, x \left(\frac{\theta - x + 1}{2\theta}\right)^{n-1} dx$$

Let $U = \frac{\theta - x + 1}{2\theta}$, $2\theta U = \theta - x + 1$ or $x = \theta + 1 - 2\theta U$

so $x = 1-\theta \to U = 1$, $x = 1+\theta \to U = 0$, $dx = -2\theta\, dU$

$$E_\theta(X_{(1)}) = \int_{+1}^{0} \frac{n}{2\theta} (\theta + 1 - 2\theta U)\, U^{n-1} (-2\theta)\, dU$$

$$= n \int_0^1 (\theta + 1 - 2\theta U)\, U^{n-1}\, dU =$$

$$n(\theta+1)\int_0^1 U^{n-1}\,dU - 2\theta n \int_0^1 U^n\, dU =$$

$$n(\theta+1) \left.\frac{U^n}{n}\right|_0^1 - 2\theta n \left.\frac{U^{n+1}}{n+1}\right|_0^1 = n(\theta+1)\frac{1}{n} - 2\theta n \frac{1}{n+1}$$

$E_\theta[X_{(1)}] = \theta + 1 - 2\theta \frac{n}{n+1}$ ($\approx 1-\theta$ as expected)

Now $f(\underline{x}) = \prod_{i=1}^{n} \frac{1}{2\theta} I[1-\theta < x_i < 1+\theta]$

$= \left(\frac{1}{2\theta}\right)^n I[1-\theta < x_{(1)} < x_{(n)} < 1+\theta]$ $\underbrace{1}_{h(\underline{x})}$

$g(T(\underline{x})|\theta) = \left(\frac{1}{2\theta}\right)^n I[\max(x_{(n)}-1, 1-x_{(1)}) < \theta]$

So $(X_{(1)}, X_{(n)})$ is suff by factorization.

(This could be used to show T_2 is not min suff, see notes 32.)

$\left(\begin{matrix} \max\left(X_{(n)}-1,\ 1-X_{(1)}\right) \\ \text{is minsuff} \end{matrix}\right)$

Now $E_\theta[a X_{(1)} + b X_{(n)} + c] = 0 \ \forall \theta$

if $a = b = 1$ and $c = -2$ since

$$1+\theta - 2\theta \frac{n}{n+1} + 1 - \theta + 2\theta \frac{n}{n+1} = 2.$$

So if $g(\underline{T}_2) = g(X_{(1)}, X_{(n)}) = X_{(1)} + X_{(n)} - 2$,
then $E_\theta[g(\underline{T}_2)] = 0 \ \forall \theta$, but

$P_\theta(g(\underline{T}_2) = 0) = 0 < 1$. So $\underline{T}_2$ is not complete

↑ $(X_{(1)}, X_{(n)})$ have a pdf so $X_{(1)} + X_{(n)} - 2$ has a pdf

Easier: $(X_{(1)}, X_{(n)})$ is not a function of $\max(X_{(n)}-1, 1-X_{(1)})$ so not min suff so not complete.

ex) Suppose $Y_1, \ldots, Y_n$ are iid $N(\mu, 1)$. Then
$T = \sum Y_i \sim N(n\mu, n)$ and

$$E_\mu\left[\sum Y_i - n\mu\right] = 0 \quad \forall \mu.$$

Can I say $g(T) = \sum Y_i - n\mu$ shows
T is not complete?
No, $g(T)$ needs to be free of
unknown parameters μ.

35) p119 * Let $Y_1, \ldots, Y_n$ have pdf or pmf $f(y|\theta)$.
A statistic $\underline{W}(\underline{Y})$ whose distr...
does not depend on θ is an <u>ancillary statistic</u>

36) **know** Basu's Lemma: Let $Y_1, \ldots, Y_n$ have pdf or pmf $f(\underline{y}|\theta)$. If $\underline{T}(\underline{Y})$ is complete sufficient, then $\underline{T}(\underline{Y})$ is independent of every ancillary statistic.

ex] For any location family, $Y_{(n)} - Y_{(1)}$ is ancillary and S^2 is ancillary.
For any scale family, $\underline{W}(\underline{Y}) = \left(\frac{Y_1}{Y_n}, \ldots, \frac{Y_{n-1}}{Y_n}\right)$ is ancillary.

37] If $\underline{T}(\underline{Y})$ is not ind of an ancillary statistic, then $T(\underline{Y})$ is not complete. See ex 4.17.

38] If $\underline{W}(\underline{Y})$ is ancillary and $\underline{T}(\underline{Y})$ complete, Basu's theorem implies $\underline{W}(\underline{Y}) \perp\!\!\!\perp \underline{T}(\underline{Y})$

without finding the joint and marginal dist's of $\underline{W}(\underline{Y})$ and $\underline{T}(\underline{Y})$.

ex] p 119 $Y_1, \ldots, Y_n$ iid $N(\mu, \sigma^2)$. For fixed σ^2, this is a location family with $\sum Y_i$ complete. Thus $\overline{Y} = \frac{1}{n}\sum Y_i \perp\!\!\!\perp S^2$ (↑ ancillary) by Basu's theorem for any $\sigma^2 > 0$ known. Thus $\overline{Y} \perp\!\!\!\perp S^2$ for $\sigma^2 > 0$ unknown.

Problems 4.31, 4.32 have solutions. end exam 2 material