Math 581 HW 1 Fall 2021. Due Thursday, Aug. 26.

Place your solutions on a separate sheet of paper. DO NOT place solutions side by side. You may use both the front and the back of each sheet.

YOU ARE BEING GRADED FOR WORK NOT JUST THE FINAL ANSWER. As a rule of thumb, you should have some idea of what you were doing, even without the book or notes. You are encouraged to form groups to discuss ideas and HW problems, but do not copy.

Exam 1 review may be useful. For the quiz, the exam 1 review and oral exam problems from the course website may be useful. 3 sheets of notes for the quiz.

1) One way to show that $A = B$ is to show i) if $\omega \in A$ then $\omega \in B$ so $A \subseteq B$, and ii) if $\omega \in B$ then $\omega \in A$ so $B \subseteq A$. Suppose for each positive integer n, $\cup_{k=1}^n A_k = \cup_{k=1}^n B_k$. Let $A = \cup_{k=1}^\infty A_k$ and $B = \cup_{k=1}^\infty B_k$. Prove $A = B$ by showing i) and ii). In probability theory, often the B_k are disjoint.

2) 2.3: a) Suppose $\Omega \in \mathcal{D}$ and $A, B \in \mathcal{D} \Rightarrow A - B = A \cap B^c \in \mathcal{D}$. Show $\mathcal{D}$ is a field. Hint: the first 3 properties of a $\sigma-$field define a field.

b) Suppose $\Omega \in \mathcal{D}$ and that $\mathcal{D}$ is closed under the formation of complements and finite disjoint unions. Show that $\mathcal{D}$ need not be a field. Hint: let $\Omega = \{1,2,3,4\}$ and $\mathcal{D} = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \Omega\}$.

3) (Similar to 4.3 c)) Suppose $(limsup_n\ A_n)^c = liminf_n\ A_n^c$ for any sequence of sets $\{A_n\}$. Show $(liminf_n\ A_n)^c = limsup_n\ A_n^c$.

4) 4.1: Suppose $A_1, A_2, ...$ decompose Ω. Hence the A_n are disjoint, $\cup_{n=1}^\infty A_n = \Omega$, and $P(A_n) > 0$. Suppose $P(B) > 0$. Derive Bayes' formula

$$P(A_k|B) = \frac{P(A_k)P(B|A_k)}{\sum_{n=1}^\infty P(A_n)P(B|A_n)}.$$

5) (Similar to 4.3a)) i) Show $(limsup_n\ A_n) \cap (limsup_n\ B_n) \supseteq limsup_n\ (A_n \cap B_n)$. Hint: $\omega \in \overline{lim}\ C_n$ means $\omega \in C_n$ infinitely often (i.o.).

ii) Show $(limsup_n\ A_n) \cup (limsup_n\ B_n) = limsup_n\ (A_n \cup B_n)$.

[Note: By i), $(limsup_n\ A_n^c) \cap (limsup_n\ B_n^c) \supseteq limsup_n\ (A_n^c \cap B_n^c)$. Taking complements of both sides shows $(liminf_n\ A_n) \cup (liminf_n\ B_n) \subseteq liminf_n\ (A_n \cup B_n)$.
By ii) $(limsup_n\ A_n^c) \cup (limsup_n\ B_n^c) = limsup_n\ (A_n^c \cup B_n^c)$. Taking complements of both sides gives $(liminf_n\ A_n) \cap (liminf_n\ B_n) = liminf_n\ (A_n \cap B_n)$.]