Math 581 HW 2 Fall 2021. Due Thursday, Sept. 2.

Exam 1 review may be useful. For quiz 2, the exam 1 review and oral exam problems from the course website may be useful. 3 sheets of notes for the quiz.

1) Let Λ be an arbitrary nonempty index set, and for $\lambda \in \Lambda$, let $\mathcal{F}_\lambda$ be a $\sigma-$field on Ω. Prove that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is a $\sigma-$field on Ω.

2) 5.3: Show that $m = E(X)$ minimizes $E[(X-m)^2]$.

3) Similar to 5.7 b). Let (a, b) be an open interval where $a = -\infty$ and $b = \infty$ are allowed. A sufficient condition for a function g to be convex on an open interval (a, b) is that $g''(x) > 0$ on (a, b). (If $(a, b) = (0, \infty)$ and g is continuous on $[0, \infty)$ and convex on $(0, \infty)$, then g is convex on $[0, \infty)$.)

Jensen's Inequality is $g(E[X]) \leq E[g(X)]$ if the expected values exist and g is convex on an interval containing the range of X. If X is a positive random variable, then the range of X is $(0, \infty)$.

i) Let $g(x) = \dfrac{1}{x^p}$ where $p > 0$ and $g : (0, \infty) \to \mathbb{R}$. Show g is convex on $(0, \infty)$.

ii) Use i) and Jensen's inequality to prove that if $p > 0$ and X is a positive random variable, then

$$E\left[\frac{1}{X^p}\right] \geq \frac{1}{(E[X])^p}.$$

(We will assume expected values exist when formulas such as the above formula are given.)

4) 4.15: Suppose $A_1, A_2, ...$ are independent. There are 4 cases for the divergence of $\sum_n P(A_n)$ and $\sum_n P(A_n^c)$. Describe the pair $P(limsup_n\, A_n)$ and $P(liminf_n A_n)$ in each case.

i) $\sum_n P(A_n) < \infty$ and $\sum_n P(A_n^c) < \infty$
ii) $\sum_n P(A_n) = \infty$ and $\sum_n P(A_n^c) = \infty$
iii) $\sum_n P(A_n) < \infty$ and $\sum_n P(A_n^c) = \infty$
iv) $\sum_n P(A_n) = \infty$ and $\sum_n P(A_n^c) < \infty$

Hint: One case is impossible, and for the other cases the probabilities are 0 or 1 by the Borel Cantelli lemmas and Theorem 4.1. Complementation may also be needed: see HW1 3) and Exam 1) review: the facts above 12), 14), and 17).