Math 581 HW 3 Fall 2021. Due Thursday, Sept. 9.

Exam 1 review may be useful. For quiz 3, the exam 1 review and oral exam problems from the course website may be useful. 5 sheets of notes for the quiz.

1) Similar to 4.14 a). Let $A_1, A_2, \dots$ be independent events.

i) Prove $P(\bigcap_{n=1}^{\infty} A_n) = \prod_{i=1}^{\infty} P(A_n)$.

ii) Prove $P(\bigcup_{n=1}^{\infty} A_n) = 1 - \prod_{i=1}^{\infty}[1 - P(A_n)]$.

Hint: for i), $B_m = \bigcap_{n=1}^{m} A_n \downarrow \bigcap_{n=1}^{\infty} A_n$ as $m \to \infty$.

2) Prove

$$X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(B_i)$$

if $X : \Omega \to \mathbb{R}$ is a random variable (measurable function and real function) and the $B_i \in \mathcal{B}(\mathbb{R})$.

3) Let μ be a measure on $(\Omega, \mathcal{F})$, and let A, B, A_i, A_n, A_k be $\mathcal{F}$ sets. Prove the following.

Hint: for a), b) and c). The proof is nearly identical to that for a probability measure, just replace P by μ.

a) Finite additivity: If $A_1, \dots, A_n$ are disjoint, then $\mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$.

b) μ is monotone: $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$.

c) If $A \subseteq B$ and $\mu(B) < \infty$, then $\mu(B - A) = \mu(B) - \mu(A)$.

d) Finite subadditivity: $\mu(\cup_{i=1}^{n} A_i) \le \sum_{i=1}^{n} \mu(A_i)$.

Hint: Let $B_1 = A_1$ and $B_k = A_k \cap A_1^c \cdots A_{k-1}^c = A_k \cap [\cup_{i=1}^{k-1} A_i]^c$. You may use the fact that the B_i are disjoint, $B_i \subseteq A_i$, and $\cup_{i=1}^{n} A_i = \cup_{i=1}^{n} B_i$, as was done for proving the analagous property for a probability measure.

e) continuity from below: If $A_n \uparrow A$ then $\mu(A_n) \uparrow \mu(A)$.

Hint: Let $B_1 = A_1$ and $B_k = A_k - A_{k-1}$. You may use the fact that the B_k are disjoint, $A_n = \cup_{i=1}^{n} A_i = \cup_{i=1}^{n} B_i$ for each n, and $A = \cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$, as was done for proving the analagous property for a probability measure.

4) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Prove that I_A is not a random variable (with respect to the probability space) if A is not a subset of $\mathcal{F}$.

Hint: see notes.