Math 581 HW 4 Fall 2021. Due Thursday, Sept. 23.

Exam 1 and 2 reviews may be useful. For quiz 4, the exam reviews and oral exam problems from the course website may be useful. 5 sheets of notes for the quiz.

Exam 1 is Thursday, September 15. **NO NOTES!**

Final: Wednesday, Dec. 8, 12:30-2:30.

1) Like part of (5.12): Let $X = \sum_{k=1}^{n} I_{A_k}$ be a simple random variable, and find $E[X/n]$.

2) 5.14: Prove that if X has nonnegative integers as values, then $E[X] = \sum_{n=1}^{\infty} P(X \geq n)$.

Hint: $E[X] = \sum_x xP(X = x) = \sum_{n=1}^{\infty} nP(X = n)$. Consider the following array, and sum on columns and sum on rows.

Table 1:

						sum
	P(X=1)	P(X=2)	P(X=3)	P(X=4)	$\cdots$	$P(X \geq 1)$
		P(X=2)	P(X=3)	P(X=4)	$\cdots$	$P(X \geq 2)$
			P(X=3)	P(X=4)	$\cdots$	$P(X \geq 3)$
				P(X=4)	$\cdots$	$P(X \geq 4)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
sum	P(X=1)	2 P(X=2)	3 P(X=3)	4 P(X=4)	$\cdots$	E(X)

3) Suppose $A_n = A$ for $n \geq 1$ where $P(A) = 0.5$. Then $A_n \to A$, $P(A_n) \to P(A) = 0.5$, $limsup_n\ A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \{\omega : \omega \in A_n \text{ for infinitely many } A_n\} = A$ and $liminf_n\ A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = \{\omega : \omega \in A_n = \text{ for all but finitely many } A_n\} = A$. It is known that $liminf\ A_n$ and $limsup\ A_n$ are tail events. Does this example contradict Kolmogorov's zero-one (0-1) law? Explain briefly.

4) Let $t : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable real function. Hence $t^{-1}(B) = \{y \in \mathbb{R} : t(y) \in B\} = B' \in \mathcal{B}(\mathbb{R})\ \forall B \in \mathcal{B}(\mathbb{R})$. Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. Prove that $Z = t(X)$ is a random variable where $Z : \Omega \to \mathbb{R}$. Hint: show $Z^{-1}(B) = X^{-1}(t^{-1}(B))$ if $B \in \mathcal{B}(\mathbb{R})$.

5) Suppose the X_n are nonnegative random variables with $lim_{n \to \infty} X_n = X$ and $lim_{n\to\infty} E(X_n) = c > 0$. What does Fatou's lemma say about these 2 quantities?