Math 581 HW 5 Fall 2021. Due Thursday, Sept. 30.

Exam 1 and 2 reviews may be useful. For quiz 5, the first two pages of exam 2 review will be useful. 5 sheets of notes for the quiz.

Final: Wednesday, Dec. 8, 12:30-2:30.

1) Suppose $limsup_n \displaystyle\int f_n du \leq liminf_n \displaystyle\int f_n du$. Does $lim_{n\to\infty} \displaystyle\int f_n d\mu$ exist? Explain briefly.

2) Let P be the uniform U(0,1) probability and let

$$X = 1I_{(0,0.75)} + 1I_{(0.5,1)}.$$

a) Find $E(X)$ using linearity: $E(\sum_{i=1}^n x_i I_{A_i}) = \sum_{i=1}^n x_i P(A_i)$.

b) Find $E(X) = \sum_x x P(X = x)$ by finding the two distinct values of x in the range of X and the two values of $P(X = x)$.

(Note: for $X = 1I_{(0,0.75)} + 1I_{(0.5,1)}$, $n = 2$, and $x_i = 1$ for $i = 1, 2$. Thus $E(X) \neq \sum_{i=1}^n x_i P(X = x_i) = 2(1)P(X = 1)$. Need the x_i to be the distinct values of the range of SRV X for $E(X) = \sum_{i=1}^n x_i P(X = x_i) = \sum_x x P(X = x)$.)

3) Fix $(\Omega, \mathcal{F}, P)$. Let the induced probability $P_X = P_F$ be $P_X(B) = P[X^{-1}(B)]$ for any $B \in \mathcal{B}(\mathbb{R})$. Show that $E[I_B(X)] = \int I_B \, dP_X$.

Hint: Find $I_B(X(\omega))$, and then take the expectation.

(Note: $E[I_B(X)] = \int I_B(x) dF(x)$. Hence $\int h(x) \, dF(x)$ and $\int h \, dP_X$ agree on indicator RVs $h = I_B$. By linearity, $\int h(x) \, dF(x)$ and $\int h \, dP_X$ agree on SRVs. By monotone passage of the limit of nonnegative SRVs, $\int h(x) \, dF(x)$ and $\int h \, dP_X$ should agree on nonnegative RVs h, and hence on general RVs h. Also, $\int h \, dP_X = E_X[h] = E_F[h]$ where $Z = h$ is a RV on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$. $E[h(X)] = \int h(X) dP = \int h(x) dF(x)$ has $W = h(X)$ a RV on $(\Omega, \mathcal{F}, P)$. Do not use these results for solving 3).)

4) Let W, X and Y be integrable.

a) Prove $|E[W]| \leq E[|W|]$.

b) Prove $|E[X] - E[Y]| \leq E[|X - Y|]$.

Hint: see Billingsley p. 210. See course notes p. 38-30 point 36) c) for a).