Math 581 HW 6 Fall 2021. Due Thursday, Oct. 7.

Exam 1 and 2 reviews may be useful. For quiz 6, the exam reviews and oral exam problems from the course website may be useful. 5 sheets of notes for the quiz.

Final: Wednesday, Dec. 8, 12:30-2:30.

1) Suppose $A_1 \supseteq A_2 \supseteq A_3 ...$ so that $A_n \downarrow A$. Prove $A = \bigcap_{n=1}^{\infty} A_n$.

2) 16.9: Let f_n be integrable and $sup_n \int f_n d\mu < \infty$. If $f_n \uparrow f$, prove that f is integrable and $\int f_n d\mu \to \int f d\mu$.

Hints: a) $0 \leq (f_n - f_1) \uparrow (f - f_1)$. Apply the MCT.

b) Let $g = f - f_1$. Then $\int g d\mu = lim_n \int (f_n - f_1) d\mu \leq sup_n \int (f_n - f_1) d\mu$. Show this implies g is integrable.

c) Then $g + f_1 = f$ is integrable.

3) 21.5. $X \sim C(0, 1)$, a Cauchy distribution with location and scale parameters 0 and 1, if the probability density function (pdf) of X is

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

for $-\infty < x < \infty$. Show $E(X)$ does not exist by showing that $E[|X|] = \infty$. Hint: $|x|f(x)$ is an even function. Thus

$$E[|X|] = \int_{-\infty}^{\infty} |x|f(x)dx = 2\int_{0}^{\infty} \frac{|x|}{\pi(1 + x^2)} dx.$$

4) 21.6 modified: Theorem 16.6: if $f_n \geq 0$, then $\int \sum_n f_n d\mu = \sum_n \int f_n d\mu$. a) Apply Theorem 16.6 to indicator RVs with $\mu = P$ to prove the first Borel Cantelli lemma.

5) 20.2: If X is a positive RV with pdf f, prove that $X^{-1} = 1/X$ has pdf

$$\frac{1}{x^2} f\left(\frac{1}{x}\right).$$

6) a) Suppose X has a mixture distribution of the U_j with probabilities π_j, and that the cdf of X is $F_X(t) = \sum_{j=1}^{J} \pi_j F_{U_j}(t)$. If $f_{U_j}(t)$ is the pdf of U_j for each j, find the pdf f_X of X.

b) Using a), show that if $E[h(X)]$ and the $E[h(U_j)]$ exist, then $E[h(X)] = \sum_{j=1}^{J} \pi_j E[h(U_j)]$.

c) Suppose X has a mixture distribution of U_1 with probability 0.95 and U_2 with probability 0.95 where $P(U_1 = 0) = 1$ and U_2 is a nonnegative random variable with $E(U_2) = 1000$ and $V(U_2) = 10000$. Find i) $E(X)$, ii) $E(X^2)$, and iii) $V(X)$.

Note: X can be the claims distribution for an insurance policy where 95% of the policy holders make no claim in the year, and 5% make a claim with a complicated nonnegative distribution U_2 where the mean and variance are known from extensive past records. Then the central limit theorem can be used to find the percentiles of $\sum X_i$ where the X_i are iid from the distribution of X.