Math 581 HW 8 Fall 2021. Due Thursday, Oct. 28.

Exam 2 and 3 reviews may be useful. For quiz 8, the exam reviews and oral exam problems from the course website may be useful. 5 sheets of notes for the quiz.

The text, Resnick, S. (1999), *A Probability Path*, Birkhauser, Boston, MA., is online from the library and looks good.

Final: Wednesday, Dec. 8, 12:30-2:30.

1) Lemma 1 (from section 27): Let $z_1, ..., z_m$ and $w_1, ..., w_m$ be complex numbers of modulus at most 1. Then $|(z_1 \cdots z_m) - (w_1 \cdots w_m)| \leq \sum_{k=1}^m |z_k - w_k|$.

Prove this lemma by induction using $(z_1 \cdots z_m) - (w_1 \cdots w_m) =$
$(z_1 - w_1)(z_2 \cdots z_m) + w_1[(z_2 \cdots z_m) - (w_2 \cdots w_m)]$. Also, the modulus $|z|$ acts much like the absolute value. Hence $|z_1 z_2| = |z_1||z_2|$, and $|z_1 + z_2| \leq |z_1| + |z_2|$.

2) The characteristic function for $Y \sim N(\mu, \sigma^2)$ is $\phi_Y(t) = \exp(it\mu - t^2\sigma^2/2)$. Let $X_n \sim N(0, n)$.

a) Prove $\phi_{X_n}(t) \to h(t)\ \forall t$ by finding $h(t)$.

b) Use a) to prove whether X_n converges in distribution. Exam 3 review 96) may be useful.

3) X has a point mass at c or X is degenerate at c if $P(X = c) = 1$.

a) Find the characteristic function of X.

b) Suppose X_n is a sequence of random variables and $\phi_{X_n}(t) \to 1\ \forall t$ as $n \to \infty$. Prove whether X_n converges in distribution.

4) Suppose X_n is a discrete random variable with $P(X_n = n) = 1/n$ and $P(X_n = 0) = (n - 1)/n$.

a) Show $X_n \xrightarrow{D} X$. (Hint: see exam 2).

b) Does $E(X_n) \to E(X)$? Explain briefly.

5) Suppose $X_1, ..., X_n$ are uncorrelated with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$. Then $E(\overline{X}_n) = \overline{\mu}_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^n \mu_i$ and $V(\overline{X}_n) = \dfrac{1}{n^2}\displaystyle\sum_{i=1}^n \sigma_i^2 \to 0$ as $n \to \infty$ Use Chebyshev's inequality (Exam 1 review 23)) to prove $(\overline{X}_n - \overline{\mu}_n) \xrightarrow{P} 0$ as $n \to \infty$.

6) If $X \sim C(0, 1)$, the Cauchy (0,1) distribution, then the characteristic function of X is $\varphi_X(t) = e^{-|t|}$.

a) If $X_1, ..., X_n$ are iid $C(0, 1)$, prove $\overline{X}_n \sim C(0, 1)$.

b) Prove $\overline{X}_n \xrightarrow{D} X$.