added 1 to each score

Math 581 Exam 1 Fall 2021 Name____________________

see 1.17

1) Suppose $A_1, A_2, \ldots$ are independent, that $\sum_n P(A_n) < \infty$ and $\sum_n P(A_n^c) = \infty$. Find $P(limsup_n\ A_n)$, find $P(liminf_n\ A_n)$ and if $P(A_n) \to c$, find c. Was independence needed? see 1.17

By the 1st Borel Cantelli lemma,

$P(\limsup_n A_n) = 0$. Thus $P(\liminf_n A_n) = 0$

and $c = 0$.

Independence was not needed (only the 1st Borel Cantelli lemma was used),

Also $\sum_n P(A_n) < \infty \Rightarrow P(A_n) \to 0$.

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1.12

2) Let $a < b$ and let $I = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] = \bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] = \bigcup_{n=m}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ $= \bigcup_{n=1}^{\infty} C_n$ where $C_n \uparrow (a,b)$, $\emptyset \subseteq A \uparrow A$

where m is the smallest positive integer such that $a + \frac{1}{m} \le b - \frac{1}{m}$ since $[c,d] = \emptyset$ if $c > d$. I is equal to an interval. ~~Find that interval.~~

$$\boxed{I = (a,b)}$$

a and b are never in $[a+\frac{1}{n}, b-\frac{1}{n}]$ for any $n \geq 1$

but $a+\varepsilon$ and $b-\varepsilon$ are eventually in I for large enough n where $\varepsilon > 0$ and $a+\varepsilon < b-\varepsilon$.

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$a < a+\frac{1}{n}$ and $b > b-\frac{1}{n}$ $\forall n \Rightarrow a, b \notin I$

3) a) Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of sets such that $P(A_n) = 0 \;\; \forall n$. Prove $P\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$.

→ for =1

$$0 \leq P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i) = 0$$

by subadditivity.

b) Let $\{B_i\}_{i=1}^{\infty}$ be a sequence of sets such that $P(B_n) = 1 \;\; \forall n$. Then $P(B_n^c) = 0 \; \forall n$, and by a), $P\left(\bigcup_{i=1}^{\infty} B_i^c\right) = 0$. Prove $P\left(\bigcap_{i=1}^{\infty} B_i\right) = 1$.

$$1 - 0 = \underbrace{P\left[\left(\bigcup_{i=1}^{\infty} B_i^c\right)^c\right]}_{= 1 - P\left(\bigcup_{i=1}^{\infty} B_i^c\right)} = P\left(\bigcap_{i=1}^{\infty} B_i\right).$$

$$P\left(\bigcap_{i=1}^{\infty} B_i\right) = 1 - P\left[\left(\bigcap_{i=1}^{\infty} B_i\right)^c\right]$$

$$= 1 - P\left[\bigcup_{i=1}^{\infty} B_i^c\right] = 1 - 0$$

$$1 - P\left[\left(\bigcup_{i=1}^{\infty} B_i^c\right)^c\right] = 0$$

$$= 1 - P\left(\bigcap_{i=1}^{\infty} B_i\right)$$

so $P\left(\bigcap_{i=1}^{\infty} B_i\right) = 1$

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1.14 4) For an arbitrary sequence of events $\{A_n\}$,

$$P(liminf_n\ A_n) \leq liminf_n\ P(A_n) \leq limsup_n\ P(A_n) \leq P(limsup_n\ A_n).$$

Also, $\lim_{n\to\infty} x_n = x$ iff $liminf_n\ x_n = limsup_n\ x_n = x$ where $x_n, x \in \mathbb{R}$.

a) Use these results to prove that if $\lim_{n\to\infty} A_n$ exists, then $P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n)$.

$$\lim_{n\to\infty} A_n = A = \liminf A_n = \limsup A_n$$

$$\therefore P(A) \leq \liminf P(A_n) \leq \limsup P(A_n) \leq P(A)$$

$$\text{or } P(A) = P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

b) ~~Let~~ Let $\{A_n\}$ be a sequence of events with the same probability $P(A_n) = p\ \forall n$. Prove $P(limsup_n\ A_n) \geq p$.

$\therefore \lim P(A_n) = p$

$$\liminf P(A_n) = \limsup P(A_n) = \lim P(A_n) = p \leq P(\limsup A_n).$$

$\Big(P(A_n) = p\ \forall n \Rightarrow \limsup P(A_n) = p$

(one cluster point of $P(A_n)$) $\Big)$

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5) Let Λ be an arbitrary nonempty index set, and for $\lambda \in \Lambda$, let $\mathcal{F}_\lambda$ be a $\sigma-$field on Ω. Prove that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is a $\sigma-$field on Ω. 1.6

i) $\Omega \in \mathcal{F}_\lambda \ \forall \lambda, \ \therefore \ \Omega \in \mathcal{F}$

ii) $A \in \mathcal{F} \Rightarrow A \in \mathcal{F}_\lambda \ \forall \lambda \Rightarrow$

$A^c \in \mathcal{F}_\lambda \ \forall \lambda \Rightarrow A^c \in \mathcal{F}$

iii) $A_1, A_2 \ldots \in \mathcal{F} \Rightarrow A_1, A_2, \ldots \in \mathcal{F}_\lambda \ \forall \lambda$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\lambda \ \forall \lambda \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$

$\therefore$ $\mathcal{F}$ is a σ-field on Ω.

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1.23

6) What is a probability space? $(\Omega, \mathcal{F}, P)$ is a probability space if Ω is a sample space, $\mathcal{F}$ is a σ-field on Ω, and P is a probability measure on $(\Omega, \mathcal{F})$.

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1.45

variant Λ nonempty prove $\left[\bigcap_{\lambda\in\Lambda} A_\lambda\right]^c = \bigcup_{\lambda\in\Lambda} A_\lambda^c$

7) Prove DeMorgan's law $\left[\bigcap_{k=n}^{N} A_k\right]^c = \bigcup_{k=n}^{N} A_k^c$ where $N \geq n$, n is a positive integer, and $N = \infty$ is allowed. Here the $A_k \subseteq \Omega$.

If $\omega \in \left[\bigcap A_k\right]^c$ then $\omega \notin \bigcap A_k$.

Thus $\omega \notin$ all A_k, $\therefore$ $\omega \in A_j^c$ for some $j \geq n$.

$\therefore$ $\omega \in \bigcup_{k=n}^{N} A_k^c$.

If $\omega \in \bigcup_{k=n}^{N} A_k^c$, then $\omega \in A_j^c$ for some $j \geq n$. (so $\omega \notin A_j$)

$\therefore$ $\omega \notin$ all A_k, $\therefore$ $\omega \notin \bigcap A_k$

$\therefore$ $\omega \in \left[\bigcap_{k=n}^{N} A_k\right]^c$.

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8) Let μ be a measure on $(\Omega, \mathcal{F})$, and let A, B, A_i be $\mathcal{F}$ sets. You may assume finite additivity: if $A_1, ..., A_n$ are disjoint, then $\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$. If $A \subseteq B$ and $\mu(B) < \infty$, prove $\mu(B - A) = \mu(B) - \mu(A)$.

$$B = \underbrace{A \cup (B-A)}_{\text{disjoint}}$$

$$\therefore \mu(B) = \mu(A) + \mu(B-A)$$

and $\mu(B-A) = \mu(B) - \mu(A)$

($\mu(B) \geq 0$ $\mu(A) \geq 0$ and $\mu(B) - \mu(A)$ is not $\infty - \infty$)

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