added 5 to each score [illegible]

Math 581 Exam 2 Fall 2021 Name___________________

3.15 1) State the central limit theorem.

Let $x_1, \dots, x_n$ be iid with $E(x_i) = \mu$ and $V(x_i) = \sigma^2$.

Then $\sqrt{n}(\bar{x}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$.

11 2) State and prove the Monotone Convergence Theorem (for random variables). 3.26

I ignore "a.e." on the proof.

If $0 \le X_n \uparrow X$ ae then $E(X_n) \uparrow E(X)$.

[illegible]

proof

$X_n \uparrow X$ $\therefore$ $E[X_n] \le E[X]$ $\forall n$

$\therefore$ $\limsup_n E(X_n) \le E[X]$.

By Fatou's lemma,

$E[X] = E[\lim X_n] = E[\liminf X_n] \le$

$\le \liminf E[X_n] \le \limsup_n E(X_n) \le E[X]$.

$\therefore$ $\lim E[X_n] = E[X]$.

$X_n \uparrow$ means $E[X_n] \le E[X_{n+k}]$ for $k \ge 0$

$\therefore$ $E[X_n] \uparrow E[X]$.

11

3.16

3) a) Suppose X has a mixture distribution of the U_j with probabilities π_j, and that the cdf of X is $F_X(t) = \sum_{j=1}^{J} \pi_j F_{U_j}(t)$. If $f_{U_j}(t)$ is the pdf of U_j for each j, find the pdf f_X of X.

$$f_X(t) = \frac{d}{dt} F_X(t) = \sum_{j=1}^{J} \pi_j \frac{d}{dt} F_{U_j}(t) = \sum_{j=1}^{J} \pi_j f_{U_j}(t)$$

b) Using a), show that if $E[h(X)]$ and the $E[h(U_j)]$ exist, then $E[h(X)] = \sum_{j=1}^{J} \pi_j E[h(U_j)]$.

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx =$$

$$\int_{-\infty}^{\infty} h(x) \sum_{j=1}^{J} \pi_j f_{U_j}(x)\,dx$$

$$= \sum_{j=1}^{J} \pi_j \int_{-\infty}^{\infty} h(x) f_{U_j}(x)\,dx$$

$$= \sum_{j=1}^{J} \pi_j E(h[U_j])$$

or $= \int_{-\infty}^{\infty} h(x)\, d\left[\sum_{j=1}^{J} \pi_j F_{U_j}(x)\right]$

$$= \sum_{j=1}^{J} \pi_j \int_{-\infty}^{\infty} h(x)\, dF_{U_j}(x)$$

$$= \sum_{j=1}^{J} \pi_j E[h(U_j)]$$

11

4.97

4) Suppose $X_1, \ldots, X_n$ are iid from a distribution with $E(X^k) = \Gamma(3-k)/6\lambda^k$ for integer $k < 4$. Recall that $\Gamma(n) = (n-1)!$ for integers $n \geq 1$ and $\Gamma(0) = 0$. Find the limiting distribution of $\sqrt{n}(\, \overline{X}_n - c\,)$ for appropriate constant c.

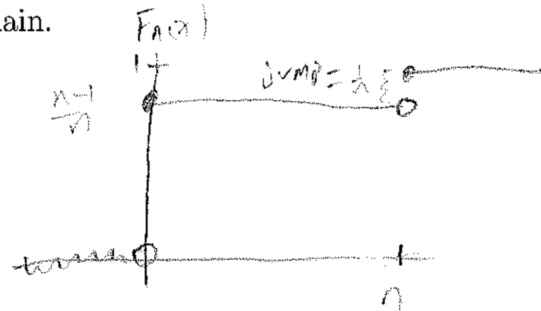
$$E(X) = E[X^1] = \frac{\Gamma(2)}{6\lambda} = \underline{\frac{1}{6\lambda}}, \quad E(X^2) = \frac{\Gamma(1)}{6\lambda^2} = \underline{\frac{1}{6\lambda^2}}$$

$$V(X) = E(X^2) - (E X)^2 = \frac{1}{6\lambda^2} - \frac{1}{36\lambda^2} = \underline{\frac{5}{36\lambda^2}}$$

$$\therefore \boxed{\sqrt{n}\left(\overline{X}_n - \frac{1}{6\lambda}\right) \xrightarrow{D} N\left(0, \frac{5}{36\lambda^2}\right)}$$

11 5) Suppose X_n is a discrete random variable with $P(X_n = n) = 1/n$ and $P(X_n = 0) = (n-1)/n$. Does $X_n \xrightarrow{D} X$? Explain.

4.98



$$F_n(x) = \begin{cases} 0 & x < 0 \\ \frac{n-1}{n} & 0 \leq x < n \\ 1 & x \geq n \end{cases} \quad \therefore F_n(x) \to F_X(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and $X_n \xrightarrow{D} X$ = pointmass at 0

11

$X = 1$ on $(0,0.6) \cup (0.7,1)$ with prob 0.9

3.17 $\quad 2$ on $(0.6,0.7)$ with prob 0.1, $\therefore E(X) = 1(.9) + 2(.1)$
$= 1.1$

6) Let P be the uniform U(0,1) probability and let $X = 1I_{(0,0.7)} + 1I_{(0.6,1)}$. Find $E(X)$.

$E(X) = 1\ P[(0,0.7)] + 1\ P[(0.6,1)]$

$= 1\ (0.7) + 1\ (1-.6) = .7 + .4$

$= \boxed{1.1}$

Q6) 25

11 4.99 (7) Let $X_n \sim \text{Poisson}(n\theta)$. Find the limiting distribution of $\sqrt{n}\left(\dfrac{X_n}{n} - \theta\right)$.

$X_n \overset{D}{=} \sum_{i=1}^{n} Y_i \quad , \quad Y_i \overset{iid}{\sim} Pois(\theta)$

$E(Y_i) = V(Y_i) = \theta$

$\therefore \sqrt{n}\left(\frac{X_n}{n} - \theta\right) \overset{D}{=} \sqrt{n}(\bar{Y}_n - \theta) \xrightarrow{D} N(0,\theta)$

by the CLT

Q7) 25

11

2.6 See 501 06/25

8) Fix $(\Omega, \mathcal{F}, P)$. Let $X : \Omega \to \mathbb{R}$. Then X is a measurable function or random variable if $X^{-1}((-\infty, t]) = \{X \leq t\} = \{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$. Prove that X is a random variable if $X^{-1}((t, \infty)) = \{X > t\} = \{\omega \in \Omega : X(\omega) > t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$.

$\{X > t\} \in \mathcal{F}\ \forall t \Rightarrow$

$\{X > t\}^c = \{X \leq t\} \in \mathcal{F}\ \forall t$

$\therefore X$ is a RV

HW6

11

9) a) Which theorem allows double integrals to be computed with iterated integrals?

3.14

Fubini's th

b) State Fatou's Lemma for random variables.

For RVs $X_n \geq 0$,

$E[\liminf X_n] \leq \liminf E(X_n)$

f_n —1

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