added 5 to each score

Zeith 50 too long

Math 581 Exam 3 Fall 2021 Name____________________

Martingales, too

e 1) Let $Y_1, ..., Y_n$ be iid Gamma(θ, θ) random variables with $E(Y_i) = \theta^2$ and $V(Y_i) = \theta^3$ where $\theta > 0$. Find the limiting distribution of $\sqrt{n}(\ \overline{Y}_n - c\ )$ for appropriate constant c.

$$\sqrt{n}\left(\overline{Y}_n - \theta^2\right) \xrightarrow{D} N(0, \theta^3)$$

4.100

Q4 HW5

7 4.82

2) Suppose $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ are iid $k \times 1$ random vectors where $E(\boldsymbol{X}_i) = (\mu_1, ..., \mu_k)^T$ and $Cov(\boldsymbol{X}_i) = (1-\alpha)\boldsymbol{I} + \alpha\boldsymbol{1}\boldsymbol{1}^T$, where $\boldsymbol{I}$ is the $k \times k$ identity matrix, $\boldsymbol{1} = (1, 1, ..., 1)^T$, and $-(k-1)^{-1} < \alpha < 1$. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{X}} - \boldsymbol{c})$ for appropriate vector $\boldsymbol{c}$. LS HW5 Q10.25

$$\boxed{\sqrt{n}\left[\overline{\underline{X}}_n - \begin{pmatrix}\mu_1\\ \vdots \\ \mu_k\end{pmatrix}\right] \xrightarrow{D} N_k\left[\underline{0}, (1-\alpha)I + \alpha \underline{1}\,\underline{1}^T\right]}$$

$$cov(X_i, X_j) = \begin{cases}\alpha & i \neq j \\ 1 & i = j\end{cases} \qquad \left(= cor(Y_i) = \frac{cov(Y_i, Y_j)}{\sqrt{V(Y_i)\,V(Y_j)}}\right)$$

$$= \begin{bmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & \alpha \cdots & \alpha \\ \alpha & \cdots & \alpha & 1 \end{bmatrix}$$

7

011225

1 90+30/90

3) Suppose X is an integrable random variable on $(\Omega, \mathcal{F}, P)$ and that σ-field $\mathbb{G} \subseteq \mathcal{F}$, $G \in \mathbb{G}$, and $A \in \mathcal{F}$. Then $E(X) = \int XdP = \int_\Omega XdP$. Use the definitions of $E(X|\mathbb{G})$ and $P(A|\mathbb{G})$ to find the following integrals.

a) $\int_G E(X|\mathbb{G})dP =$ $\boxed{\int_G X \, dP}$ def −4 if wrong

5.9

$= E[X I_g]$

b) $E[E(X|\mathbb{G})] = \int_\Omega E(X|\mathbb{G})dP =$ $\int_\Omega X \, dP = \boxed{E[X]}$

iterated expectations

c) $\int_G E(I_A|\mathbb{G})dP =$ $\int_G I_A \, dP = \int I_A I_G \, dP$

$= \int I_{A \cap G} \, dP = \boxed{P(A \cap G)}$

d) $\int_G P(A|\mathbb{G})dP =$ $\boxed{P(A \cap G)}$ def −4 if wrong

e) $\int_\Omega P(A|\mathbb{G})dP =$ $= P(A \cap \Omega) = \boxed{P(A)}$

30

$\varphi_{t^T X_n}(y) \to \varphi_{t^T X}(y)\ \forall y \qquad \varphi_{t^T X_n}(y) = E[e^{i y t^T X_n}] = \varphi_{X_n}(y t)$

$\varphi_{t^T X}(y) = \varphi_X(y t)$ take $y = 1$ to get $\varphi_{X_n}(t) \to \varphi_X(t)$

4) Let $\boldsymbol{X}$ be a $k \times 1$ random vector and $\boldsymbol{X}_n$ be a sequence of $k \times 1$ random vectors and suppose that

4.81

LSHW5

$$\boldsymbol{t}^T \boldsymbol{X}_n \xrightarrow{D} \boldsymbol{t}^T \boldsymbol{X}$$

~~$t = (0, \ldots, 0, 1, 0, \ldots, 0)^T \Rightarrow X_i \xrightarrow{D} X_i$~~

marginal does not imply joint

for all $\boldsymbol{t} \in \mathbb{R}^k$. Does $\boldsymbol{X}_n \xrightarrow{D} \boldsymbol{X}$? Explain briefly.

Yes $\underline{x}_n \xrightarrow{D} \underline{x}$ iff $\underline{t}^T \underline{x}_n \xrightarrow{D} \underline{t}^T \underline{x}\ \ \forall \underline{t} \in \mathbb{R}^k$

by the Cramér Wold device

Q10 Z5

≈ Q9 ll

5) If Y is Cauchy $C(\mu, \sigma)$, then the characteristic function of Y is $\varphi_Y(t) = \exp(it\mu - |t|\sigma)$ for all $t \in \mathbb{R}$. If $Y_1, \ldots, Y_n$ are iid Cauchy $C(\mu, \sigma)$, does $\overline{Y}_n \xrightarrow{D} W$ for some random variable W? Prove or disprove.

4.80

LS HW4

$\varphi_{\sum_{i=1}^n Y_i}(t) = \prod_{i=1}^n \varphi_{Y_i}(t) = [\exp(it\mu - |t|\sigma)]^n$

$= \exp(itn\mu - |t|n\sigma)$.

$\varphi_{\overline{Y}_n}(t) = \varphi_{\frac{\sum Y_i}{n}}(t) = E[e^{it\frac{1}{n}\sum_{i=1}^n Y_i}] = \varphi_{\sum_{i=1}^n Y_i}(\frac{t}{n})$

$= \exp(i\frac{t}{n}n\mu - \frac{|t|}{n}n\sigma) =$

$\exp(it\mu - |t|\sigma) = \varphi_Y(t)$.

$\therefore\ \overline{Y}_n \xrightarrow{D} Y \sim C(\mu, \sigma)$

by the Continuity Thm

$\varphi_{\overline{Y}_n}(t)$

$= E(e^{i\frac{t}{n}\sum Y_i})$

$= \varphi_{\sum Y_i}(\frac{t}{n})$

$= \prod_{i=1}^n \varphi_{Y_i}(\frac{t}{n})$

$= [\exp(i\frac{t}{n}\mu - \frac{|t|}{n}\sigma)]^n$

$= \exp(it\mu - |t|\sigma)$

$\forall n$ and $\forall t$

$\therefore \varphi_{\overline{Y}_n}(t) \to \varphi_Y(t)\ \forall t$

$\therefore \overline{Y}_n \xrightarrow{D} Y$

x	0	$\sqrt{n}$
$P(X_n=x)$	$1-\frac{1}{n}$	$\frac{1}{n}$
x^2	0	n

4.3.02

6) Let $X_n = \sqrt{n}$ with probability $1/n$ and $X_n = 0$ with probability $1 - 1/n$. ($X_n = \sqrt{n} I_{[0,1/n]}$ wrt U(0,1) probability.)

e a) Prove that $X_n \xrightarrow{1} 0$.

$$E(X_n) = E[|X_n|] = E[|X_n - 0|] = 0 + \sqrt{n}\,\tfrac{1}{n} = \tfrac{1}{\sqrt{n}} \to 0$$

as $n \to \infty$. $\therefore X_n \xrightarrow{1} 0$.

X where $P(X=0)=1$

a 4.02.25

b) Does $X_n \xrightarrow{2} 0$? Prove or disprove.

$$E(X_n^2) = E[|X_n - 0|^2] = 0 + \tfrac{n}{n} = 1 \quad \forall n$$

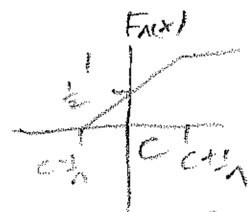
$\therefore E[|X_n - 0|^2] \to 1 \neq 0$

so $X_n \not\xrightarrow{2} 0$.

4.101 like final #11

14 → 7) Suppose $X_n \sim U(c - 1/n, c + 1/n)$. Does $X_n \xrightarrow{D} X$ for some random variable X? Prove or disprove.

(If $Y \sim U(\theta_1, \theta_2)$ then $F_Y(y) = \frac{y-\theta_1}{\theta_2-\theta_1}$, $\theta_1 \le y \le \theta_2$.)



$$F_n(x) = \begin{cases} 0, & x \le c - \frac{1}{n} \\ \frac{n}{2}\left(x - c + \frac{1}{n}\right), & c - \frac{1}{n} < x < c + \frac{1}{n} \\ 1, & x \ge c + \frac{1}{n} \end{cases} \to G(x) = \begin{cases} 0 & x < c \\ \frac{1}{2} & x = c \\ 1 & x > c \end{cases}$$

$\therefore X_n \to X$ where $P(X=c)=1$, $F_X(x) = \begin{cases} 0 & x < c \\ 1 & x \ge c \end{cases}$

$$F_n(x) = \frac{x - c + \frac{1}{n}}{2/n} = \frac{nx}{2} - \frac{cn}{2} + \frac{1}{2}, \quad c - \tfrac{1}{n} \le x \le c + \tfrac{1}{n}$$

7

−4 if they say $X_n \not\to X$

$P\{\|\overline{X}-\underline{c}\| \geq \varepsilon\} \leq \frac{E\|\overline{X}-\underline{c}\|}{\varepsilon} \to 0$ if $\underline{c}=\underline{0}$

4.102

8) Suppose $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ are iid with $E(\boldsymbol{X}_i) = \boldsymbol{0}$ but $Cov(\boldsymbol{X}_i)$ does not exist. Does $\overline{\boldsymbol{X}}_n \xrightarrow{P} \boldsymbol{c}$ for some constant vector $\boldsymbol{c}$? Explain briefly.

$\overline{X}_n \xrightarrow{P} \underline{0} = E(\underline{X}_i)$ by WLLN

7

9) Suppose $\boldsymbol{X}_n \xrightarrow{D} \boldsymbol{X}$ and $\boldsymbol{Y}_n - \boldsymbol{X}_n \xrightarrow{P} \boldsymbol{0}$. Does $\boldsymbol{Y}_n \xrightarrow{D} \boldsymbol{W}$ for some random vector $\boldsymbol{W}$? [Hint: $\boldsymbol{Y}_n = \boldsymbol{X}_n + (\boldsymbol{Y}_n - \boldsymbol{X}_n)$.]

4.103

$\underline{Y_n} = \underline{X_n} + (\underline{Y_n} - \underline{X_n}) \xrightarrow{D} \underline{X} + 0 = \underline{X}$

so $\underline{Y_n} \xrightarrow{D} \underline{X}$ by Slutsky's thm

7

4.17

10) Let $P(X_n = n) = 1$. Does X_n converge in distribution? Explain.

no, $F_{X_n}(x) = \begin{cases} 1 & x \geq n \\ 0 & x < n \end{cases} \to G(x) \equiv 0$
which is contin but not a cdf

see Q6 2.5

7

$m_{X_n}(t) = e^{tn} \to \begin{cases} 0 & t < 0 \\ 1 & t = 0 \\ \infty & t > 0 \end{cases}$ inconclusive

$\varphi_{X_n}(t) = e^{itn}$ does not converge

-4 if yes