15 pnts 20 each

added 10 to each score because of #4

Math 581 Final Fall 2021 Name___________________

1, 2, 1) Prove $(limsup_n A_n)^c = liminf_n B_n$ and find B_n.

$$\left[\limsup A_n\right]^c = \Big[\bigcap_{n=1}^{\infty} \underbrace{\Big[\bigcup_{k=n}^{\infty} A_k\Big]}_{C_n}\Big]^c = \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k=n}^{\infty} A_k^c} = \liminf \underbrace{A_n^c}$$

$$= \bigcup_{n=1}^{\infty} \underbrace{\Big[\bigcup_{k=n}^{\infty} A_k\Big]^c}_{C_n^c} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c, \qquad \boxed{B_n = A_n^c}$$

$\Big([\bigcap C_n]^c = \bigcup C_n^c$ and $[\bigcup A_k]^c = C_n^c = \bigcap A_k^c$

by De Morgan's laws twice $\Big)$

20 2) Simplify the following sets. Answers might be $(a,b), [a,b), (a,b], [a,b], [a,a] = \{a\}, (a,a) = \emptyset$. where $a < b$. See 1.17

A= i) $\bigcup_{n=1}^{\infty} \underbrace{\left(a, b - \frac{1}{n}\right]}_{A_n} =$ (a, b) $= \bigcup_{n \in \mathbb{N}} (a, b - \frac{1}{n}]$

$b \notin A_n \ \forall n$

but $b - \delta \in A$ eventually

if $0 < \delta$ and $b - \delta > a$

D= ii) $\bigcup_{n=1}^{\infty} \underbrace{\left[a, b - \frac{1}{n}\right]}_{A_n} =$ $[a, b)$ $= \bigcup_{n \in \mathbb{N}} [a, b - \frac{1}{n}]$

$a \in D_n \ \forall n$

$b \notin D_n \ \forall n$

but $b - \delta \in D$ eventually

if $0 < \delta$ and $b - \delta > a$

-8 if wrong

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Pm 2.21

Q 3) Fix $(\Omega, \mathcal{F}, P)$. For a random variable X, prove that the induced probability $P_X(B) = P[X^{-1}(B)]$ for $B \in \mathcal{B}(\mathbb{R})$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. You may use without proof i) $X^{-1}(\mathbb{R}) = \Omega$, ii) $X^{-1}(\emptyset) = \emptyset$, and iii) $X^{-1}(\bigcup_{i=1}^{\infty} B_i) = \bigcup_{i=1}^{\infty} X^{-1}(B_i)$.
iv) If A and C are disjoint, then $X^{-1}(A)$ and $X^{-1}(C)$ are disjoint.

P1) Let $B \in \mathcal{B}(\mathbb{R})$. Then

$P_X(B) = P(X^{-1}(B))$ so

$0 \leq P_X(B) \leq 1$.

P2) $P_X(\mathbb{R}) = P(X^{-1}(\mathbb{R})) = P(\Omega) = 1$

$P_X(\emptyset) = P(X^{-1}(\emptyset)) = P(\emptyset) = 0$

P3) Let $\{B_i\}_{i=1}^{\infty}$ be disjoint $\mathcal{B}(\mathbb{R})$ sets.

Then $P_X\left(\bigcup_{i=1}^{\infty} B_i\right) = P\left[X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right]$

$= P\left[\bigcup_{i=1}^{\infty} X^{-1}(B_i)\right] =$

$\sum_{i=1}^{\infty} P\left[X^{-1}(B_i)\right] = \sum_{i=1}^{\infty} P_X(B_i)$.

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Need $X^{-1}(C) \cap X^{-1}(D) = X^{-1}(C \cap D)$ as above.

Q 4) Let $\sigma(X) =$ the collection $\{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$. Prove that $\sigma(X)$ is a σ-field. You may use without proof i) $X^{-1}(\mathbb{R}) = \Omega$, ii) $X^{-1}(\emptyset) = \emptyset$, iii) $X^{-1}(\bigcup_{i=1}^{\infty} B_i) = \bigcup_{i=1}^{\infty} X^{-1}(B_i)$, and iv) if A and C are disjoint, then $X^{-1}(A)$ and $X^{-1}(C)$ are disjoint, and

v) $[X^{-1}(B)]^c = X^{-1}(B^c)$.

$\sigma 1)$ $X^{-1}(\mathbb{R}) = \Omega \in \sigma(X)$

$\sigma 2)$ Let $A \in \sigma(X)$. Then

$A = X^{-1}(B)$ for some $B \in \mathbb{B}(\mathbb{R})$.

$\therefore$ $A^c = [X^{-1}(B)]^c = X^{-1}(B^c) \in \sigma(X)$

$\sigma 3)$ Let $\{A_i\}_{i=1}^{\infty} \in \sigma(X)$, Then

$A_i = X^{-1}(B_i)$ for some $B_i \in \mathbb{B}(\mathbb{R})$,

$\therefore$ $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} X^{-1}(B_i) = X^{-1}\left[\bigcup_{i=1}^{\infty} B_i\right] \in \sigma(X)$

$\square$

~~I have an exact proof without [illegible] the notes.~~

took off add back to final score

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not needed
2nd ⇒ P(limsup A_n) = 1

1st ⇒ P(limsup A_n^c) = 0

Q 5) Suppose $A_1, A_2, \ldots$ are independent, that $\sum_n P(A_n) = \infty$ and $\sum_n P(A_n^c) < \infty$. Find $P(limsup_n\, A_n)$, find $P(liminf_n\, A_n)$ and if $P(A_n) \to c$, find c.

See 1.17

By 1st Borel cantelli lemma

$$P[\limsup A_n^c] = 0 = P[(\liminf A_n)^c]$$

$$P[\liminf A_n] = 1 - 0 = \underline{1} \leq P(\limsup_n A_n) = \boxed{1 = c}$$

$\therefore$ $P(A_n) \to 1$

(By 2nd Borel cantelli lemma, $P(\limsup A_n) = 1$.)

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Q

6) Prove the following theorem: If $X_n \geq 0$, then $E[\sum_{n=1}^{\infty} X_n] = \sum_{n=1}^{\infty} E[X_n]$.
Hint: Let $0 \leq Y_m = \sum_{n=1}^{m} X_n$ and let $0 \leq Y = \sum_{n=1}^{\infty} X_n$.

$$0 \leq Y_m = \sum_{n=1}^{m} X_n \uparrow Y = \sum_{n=1}^{\infty} X_n .$$

$$\therefore \lim_{m\to\infty} E(Y_m) = \lim_{m\to\infty} E\left[\sum_{n=1}^{m} E(X_n)\right] = \sum_{n=1}^{\infty} E(X_n) = E(Y)$$

by MCT

$$= E\left[\sum_{n=1}^{\infty} X_n\right].$$

E2J25

$\overline{20}$

7) Let $X_n \sim$ binomial(n, p). Find the limiting distribution of $\sqrt{n}\left(\frac{X_n}{n} - p\right)$. Note that $X_n \stackrel{D}{=} \sum_{i=1}^{n} Y_i$ where the $Y_i \sim$ binomial(1,p) are iid.

Part of a ch4 Q problem 4.6

$$E(Y_i) = p \qquad V(Y_i) = p(1-p)$$

$$\therefore \sqrt{n}\left(\frac{X_n}{n} - p\right) \stackrel{D}{=} \sqrt{n}(\overline{Y}_n - p) \xrightarrow{D} N[0, p(1-p)]$$

by the CLT

$np(1-p) \to \infty$

wrong, eg P)

-7

E2J25

$\overline{20}$

4.9.4 8) Let $W \sim N(\mu_W, \sigma_W^2)$ and let $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The characteristic function of W is

$$\varphi_W(y) = E(e^{iyW}) = \exp\left(iy\mu_W - \frac{y^2}{2}\sigma_w^2\right).$$

Suppose $W = \boldsymbol{t}^T\boldsymbol{X}$. Then $W \sim N(\mu_W, \sigma_W^2)$. Find μ_W and σ_W^2. Then the characteristic function of $\boldsymbol{X}$ is

$$\varphi_{\boldsymbol{X}}(\boldsymbol{t}) = E(e^{i\boldsymbol{t}^T\boldsymbol{X}}) = \varphi_W(1).$$

Use these results to find $\varphi_{\boldsymbol{X}}(\boldsymbol{t})$.

LS 0005 like 582 E2024

$\mu_W = E[\underline{t}^T \underline{x}] = \underline{t}^T E(\underline{x}) = \underline{t}^T \underline{\mu}$

$\sigma_W^2 = V(\underline{t}^T \underline{x}) = Cov(\underline{t}^T \underline{x}) = \underline{t}^T \underline{\Sigma}\, \underline{t}$

$$\boxed{\varphi_{\underline{x}}(\underline{t}) = \exp\left(i\, \underline{t}^T \underline{\mu} - \frac{1}{2}\, \underline{t}^T \underline{\Sigma}\, \underline{t}\right).}$$

$= \exp\left(i \mu_W - \frac{1}{2}\sigma_W^2\right)$

E3025

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4.96 9) Suppose $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ are iid $k \times 1$ random vectors where $E(\boldsymbol{X}_i) = \boldsymbol{1} = (1, ..., 1)^T$ and $Cov(\boldsymbol{X}_i) = \boldsymbol{I}_k = diag(1, ..., 1)$, the $k \times k$ identity matrix. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{X}} - \boldsymbol{c})$ for appropriate vector $\boldsymbol{c}$.

LS 0005

$$\boxed{\sqrt{n}(\overline{\underline{x}} - \underline{1}) \xrightarrow{D} N_K(\underline{0}, I_K)}$$

or $\to$

by MCLT

E3025

582 E2024

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5.4 10) Suppose X is an integrable random variable on $(\Omega, \mathcal{F}, P)$ and that σ-field $\mathbb{G} \subseteq \mathcal{F}$, $G \in \mathbb{G}$, and $A \in \mathcal{F}$. Then $E(X) = \int X dP = \int_\Omega X dP$. Use the definitions of $E(X|\mathbb{G})$ and $P(A|\mathbb{G})$ to find the following integrals. Simplify.

a) $E[E(X|\mathbb{G})] = \int_\Omega E(X|\mathbb{G}) dP =$ $\int_\Omega X\,dP = \boxed{E(X)}$

b) $\int_\Omega P(A|\mathbb{G}) dP =$ $= P(A \cap \Omega) = \boxed{P(A)}$

E3 #25

Like Exam 3 #7 old 2021

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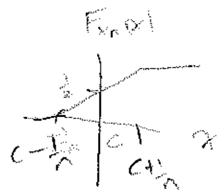
LSQ 2022

11) Suppose $X_n \sim U(c-\frac{1}{n}, c+\frac{1}{n})$

$$F_{X_n}(x) = \begin{cases} 0, & x \leq c - \frac{1}{n} \\ \frac{n}{2}(x - c + \frac{1}{n}), & c - \frac{1}{n} < x < c + \frac{1}{n} \\ 1, & x \geq c + \frac{1}{n}. \end{cases}$$



Does $X_n \xrightarrow{D} X$ for some random variable X? Prove or disprove. If $X_n \xrightarrow{D} X$, find X.

4.46

E2 #25

$$F_{X_n}(x) \to G(x) = \begin{cases} 0 & x < c \\ \frac{1}{2} & x = c \\ 1 & x > c \end{cases}$$

5 omitted ~ -5

$\therefore X_n \xrightarrow{D} X$ where $P(X=c) = 1$

$$\text{and } F_X(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

≤ -5

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2 got it part of Q

see 501 Q6d25

12) Fix $(\Omega, \mathcal{F}, P)$. Let $X : \Omega \to \mathbb{R}$. Then X is a measurable function or random variable if $X^{-1}((-\infty, t]) = \{X \leq t\} = \{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$. If X and Y are random variables, prove that $W = \max(X, Y)$ is a random variable.

2.5

$\underbrace{\{\max(X,Y) \leq t\} = \{X \leq t\} \cap \{Y \leq t\}}_{\text{both are } \leq t} \in \mathcal{F} \quad \forall t$

or −1

−9 if no attempt

−8 if U

E2d25

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13) Fix $(\Omega, \mathcal{F}, P)$. If $A \in \mathcal{F}$ is an event, prove $P(A^C) = 1 - P(A)$. You may assume finite additivity.

1.22

$\Omega = \underbrace{A \cup A^c}_{\text{disjoint}}$

Hence $P(\Omega) = P(A) + P(A^c) = 1$

or $P(A^c) = 1 - P(A)$

E1d23

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