Math 581 Quiz 10 Fall 2021 Name____________________

4.34

1) Suppose $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ are iid 3×1 random vectors from a multinomial distribution with

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$$E(\boldsymbol{X}_i) = \begin{bmatrix} m\rho_1 \\ m\rho_2 \\ m\rho_3 \end{bmatrix} \text{ and } \operatorname{Cov}(\boldsymbol{X}_i) = \begin{bmatrix} m\rho_1(1-\rho_1) & -m\rho_1\rho_2 & -m\rho_1\rho_3 \\ -m\rho_1\rho_2 & m\rho_2(1-\rho_2) & -m\rho_2\rho_3 \\ -m\rho_1\rho_3 & -m\rho_2\rho_3 & m\rho_3(1-\rho_3) \end{bmatrix}$$

where m is a known positive integer and $0 < \rho_i < 1$ with $\rho_1 + \rho_2 + \rho_3 = 1$. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{X}} - \boldsymbol{c})$ for appropriate vector $\boldsymbol{c}$.

$\sqrt{n}(\bar{X}_n - E(X_i)) \xrightarrow{D} N_3(\underline{0}, \operatorname{cov}(X_i))$ by MCLT

or $\sqrt{n}\left(\bar{X}_n - \begin{pmatrix} m\rho_1 \\ m\rho_2 \\ m\rho_3 \end{pmatrix}\right) \xrightarrow{D} N_3\left(\underline{0}, \begin{bmatrix} m\rho_1(1-\rho_1) & -m\rho_1\rho_2 & -m\rho_1\rho_3 \\ -m\rho_1\rho_2 & m\rho_2(1-\rho_2) & -m\rho_2\rho_3 \\ -m\rho_1\rho_3 & -m\rho_2\rho_3 & m\rho_3(1-\rho_3) \end{bmatrix}\right)$

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2) Suppose X_n are random variables with characteristic functions $\varphi_{X_n}(t)$, and that $\varphi_{X_n}(t) \to e^{itc}$ for every $t \in \mathbb{R}$ where c is a constant. Does $X_n \xrightarrow{D} X$ for some random variable X? Explain briefly. Hint: Is the function $g(t) = e^{itc}$ continuous as $t = 0$? Is there a random variable that has characteristic function $g(t)$?

5.22 HW3

4.82 Yes $g(t)$ is continuous at $t=0$

or Yes $e^{itc} = \varphi_X(t)$ where $P(X=c)=1$

$X \sim N(c, 0)$

$X_n \xrightarrow{D} c$

$X_n \xrightarrow{D} X$ where $P(X=c)=1$

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$|X_n|^r = X_n^r$

X_n^r	0	n^r
X_n	0	n
	$1-\frac{1}{n^r}$	$\frac{1}{n^r}$

4.78

3) Let $P(X_n = 0) = 1 - 1/n^r$ and $P(X_n = n) = 1/n^r$ where $r > 0$.

20 a) Prove that X_n does not converge in rth mean to 0. Hint: Find $E[|X_n|^r]$.

LS HW3 #6

$$E[X_n^r] = 0 + n^r \frac{1}{n^r} = 1$$

$$\therefore E[|X_n - 0|^r] \not\to 0 \text{ as } n \to \infty.$$

$\to$ 10 b) Does $X_n \xrightarrow{D} X$ for some random variable X? Prove or disprove.

$F_{X_n}(t) = \begin{cases} 0 & t < 0 \\ 1-\frac{1}{n^r} & 0 \le t < n \\ 1 & t \ge n \end{cases}$

$\to F_X(t) = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0 \end{cases}$ where $P(X=0) = 1$

or $P[|X_n| \ge \varepsilon] = \begin{cases} P(X_n = n) = \frac{1}{n^r} & 0 < \varepsilon \le n \\ 0 & \varepsilon > n \end{cases}$

$\therefore X_n \xrightarrow{P} 0 \Rightarrow X_n \xrightarrow{D} 0$

or pmf $p_{X_n}(x) \to p_X(x)$ $\forall x$ where $P(X=0) = 1$ so $X_n \xrightarrow{D} 0$

or ch fn $\varphi_{X_n}(t) = e^{it0}(1-\frac{1}{n^r}) + e^{itn}\frac{1}{n^r} \to e^{it0} = 1 = \varphi_X(t)$ where $P(X=0)=1$

30 $\to$ 4) Let Y_n and Y be random variables such that $Y_n = Y$ with probability $1 - p_n$ and $Y_n = n$ with probability p_n where $p_n \to 0$. Prove or disprove: $Y_n \xrightarrow{D} Y$.

4.109

If $Y_n = Y$, then $|Y_n - Y| = 0 < \varepsilon$. Let $\varepsilon > 0$.

$$P[|Y_n - Y| \ge \varepsilon] \le p_n \to 0$$

(equality holds iff $|Y-n| \ge \varepsilon$)

$P[|Y_n - Y| < \varepsilon] \ge 1 - p_n \to 1$

could have $E|Y| = \infty$

$$\therefore Y_n \xrightarrow{P} Y \quad \therefore Y_n \xrightarrow{D} Y$$

$P(W_n = n) = 1$

$$\text{or } \varphi_{Y_n}(t) = \underbrace{(1-p_n)E[e^{itY}] + p_n E e^{itn}}_{\text{mixture dist}}$$

($E e^{itn} = \varphi_{W_n}(t)$)

$$= (1-p_n)\varphi_Y(t) + p_n \varphi_{W_n}(t) \to \varphi_Y(t) \quad \therefore Y_n \xrightarrow{D} Y$$

or $F_{Y_n}(t) = (1-p_n)F_Y(t) + p_n F_{W_n}(t)$ where $P(W_n = n) = 1$

$F_{W_n}(t) = \begin{cases} 0, & t < n \\ 1, & t \ge n \end{cases}$

$\to F_Y(t)$

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Does $Y_n \xrightarrow{as} Y$? $\lim Y_n = Y$ everywhere so yes