Martingales, too

Math 581 Quiz 11 Fall 2021 Name____________________

1) Suppose $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ are iid $k \times 1$ random vectors where $E(\boldsymbol{X}_i) = (\mu_1, ..., \mu_k)^T$ and $Cov(\boldsymbol{X}_i) = diag(\sigma_1^2, ..., \sigma_k^2)$, a diagonal $k \times k$ matrix with jth diagonal entry σ_j^2. The nondiagonal entries are 0. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{X}} - \boldsymbol{c})$ for appropriate vector $\boldsymbol{c}$.

$$\sqrt{n}\left[\overline{\underset{\sim}{X}} - \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix}\right] \xrightarrow{D} N_k\left[\underline{0}, diag(\sigma_1^2, \ldots, \sigma_k^2)\right]$$

$$\sqrt{n}\left[\overline{\underset{\sim}{X}} - E\underset{\sim}{X}_i\right] \xrightarrow{D} N_k\left[\underline{0}, Cov(\underset{\sim}{X}_i)\right]$$

2) a) Find $\int I_A dp$.

$= P(A)$

b) Find $\int_G I_A dp$.

$= \int I_A I_G \, dP = \int I_{A \cap G} \, dP$

$= P(A \cap G)$

Also $\underset{\sim}{a}^T \underset{\sim}{Y} \sim N(\underset{\sim}{a}^T \underset{\sim}{\mu}, \underset{\sim}{a}^T \Sigma \underset{\sim}{a})$ and $\varphi_W(y) = \exp[i y E(W) - \frac{y^2}{2} V(W)]$

$= \exp[i y \, a^T \mu - \frac{y^2}{2} a^T \Sigma a]$

4.1109

3) If $\boldsymbol{X} \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then the characteristic function of $\boldsymbol{X}$ is

$$\varphi_{\boldsymbol{X}}(\boldsymbol{t}) = \exp\left(i\boldsymbol{t}^T\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{t}^T\boldsymbol{\Sigma}\boldsymbol{t}\right)$$

$\varphi_{a^T X}(y) = E[e^{i y a^T X}]$
$= \varphi_X(y a)$

modified
Good
LG E2022

for $\boldsymbol{t} \in \mathbb{R}^k$. Let $\boldsymbol{a} \in \mathbb{R}^k$ and find the characteristic function of $\boldsymbol{a}^T\boldsymbol{X} = \varphi_{\boldsymbol{a}^T\boldsymbol{X}}(y) = E[\exp(i\ y\ \boldsymbol{a}^T\boldsymbol{X})] = \varphi_{\boldsymbol{X}}(\boldsymbol{t})$ for any $y \in \mathbb{R}$ and some vector $\boldsymbol{t} \in \mathbb{R}^k$ that depends on y.

$$\varphi_{\underset{\sim}{a}^T X}(y) = \varphi_{\underset{\sim}{X}}(y \underset{\sim}{a}) = \exp[i\, y\, \underset{\sim}{a}^T \underset{\sim}{\mu} - \tfrac{1}{2} y a^T \Sigma a\, y]$$

$$= \exp[i\, y\, \underset{\sim}{a}^T \underset{\sim}{\mu} - \tfrac{1}{2} y^2 \underset{\sim}{a}^T \Sigma \underset{\sim}{a}]$$

$\overline{10}$ 4.1106

4) Suppose $\boldsymbol{X} = \boldsymbol{c}$ for some constant vector $\boldsymbol{c} \in \mathbb{R}^k$. Prove $\boldsymbol{c} \sim N_k(\boldsymbol{c}, \boldsymbol{0})$ where $\boldsymbol{0}$ is the $k \times k$ matrix of zeroes. Hint: find the characteristic function of $\boldsymbol{X}$ where $P(\boldsymbol{X} = \boldsymbol{c}) = 1$, and compare to the characteristic function given in problem 3).

$$\varphi_{\underset{\sim}{X}}(t) = E[e^{i \underset{\sim}{t}^T \underset{\sim}{X}}] = \sum_{\underset{\sim}{x}} e^{i \underset{\sim}{t}^T \underset{\sim}{x}} P[\underset{\sim}{X} = \underset{\sim}{x}] = e^{i \underset{\sim}{t}^T \underset{\sim}{c}}$$

$$= \exp[i \underset{\sim}{t}^T \underset{\sim}{c} - \tfrac{1}{2} \underset{\sim}{t}^T 0 \underset{\sim}{t}]$$

$$\text{so } \underset{\sim}{X} = \underset{\sim}{c} \sim N_k[\underset{\sim}{c}, 0]$$

$\overline{10}$