Math 581 Quiz 6 Fall 2021 Name______________________

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1) Suppose X has a mixture distribution of the U_j with probabilities π_j, and that the cdf of X is $F_X(t) = \sum_{j=1}^J \pi_j F_{U_j}(t)$. If each U_j is a discrete RV with probability mass function (pmf) $p_{U_j}(t)$, then X is a discrete RV with pmf

$$p_X(t) = \sum_{j=1}^{J} \pi_j \, p_{U_j}(t).$$

Use the pmf $p_X(t)$ to show that if $E[h(X)]$ and the $E[h(U_j)]$ exist, then $E[h(X)] = \sum_{j=1}^J \pi_j E[h(U_j)]$.

$$E[h(X)] = \sum_t h(t) p_X(t) =$$

$$\sum_t h(t) \sum_{j=1}^J \pi_j p_{U_j}(t)$$

$$= \sum_{j=1}^J \pi_j \sum_t h(t) p_{U_j}(t)$$

$$= \sum_{j=1}^J \pi_j E[h(U_j)]$$

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2) Fix $(\Omega, \mathcal{F}, P)$ and let X be a RV. What is the induced probability $P_X(B)$ for $B \in \mathcal{B}(\mathbb{R})$?

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$$P_X(B) = P[X^{-1}(B)]$$

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3) Prove **one** of the following: a) the Monotone Convergence Theorem for RVs, b) If $X_n \geq 0$, then $E[\sum_{n=1}^{\infty} X_n] = \sum_{n=1}^{\infty} E[X_n]$, or c) Lebesgue's Dominated Convergence Theorem for RVs. State which result, a), b) or c) that you are proving.

a) $X_n \uparrow X \quad \therefore E[X_n] \leq E(X) \quad \forall n$

$\therefore \limsup_n E(X_n) \leq E(X)$

By Fatou's lemma, $E(X) = E(\lim X_n) = E[\liminf X_n]$

$\leq \liminf E(X_n) \leq \limsup_n E(X_n) \leq E[X]$

$\therefore \lim E[X_n] = E[X]$. $X_n \uparrow$ means

$E[X_n] \leq E[X_{n+k}]$ for $k \geq 0$. $\therefore E[X_n] \uparrow E[X]$.

b) $0 \leq Y_m = \sum_{n=1}^{m} X_n \uparrow Y = \sum_{n=1}^{\infty} X_n$.

$\therefore \lim_{m \to \infty} E(Y_m) = \lim_{m \to \infty} \sum_{n=1}^{m} E(X_n) = \sum_{n=1}^{\infty} E(X_n)$

$\underset{\text{by MCT}}{=} E(Y) = E[\sum_{n=1}^{\infty} X_n]$.

c) $Y - X_n \geq 0$, $Y + X_n \geq 0$. $\therefore E[Y] - E[X] = E[Y-X]$

$= E[\liminf (Y - X_n)] \underset{\text{Fatou}}{\leq} \liminf E(Y - X_n) = E(Y) - \limsup E[X_n]$

$\lim (Y - X_n) = \liminf (Y - X_n) = Y - X \qquad \therefore -E[X] \leq -\limsup E[X_n]$

$E[X] \geq \limsup E[X_n]$

$E[Y] + E[X] = E[Y+X] = E[\liminf (Y + X_n)] \leq \liminf E(Y + X_n)$

$= E[Y] + \liminf E[X_n]$.

$\therefore E[X] \leq \liminf E[X_n] \leq \limsup E(X_n) \leq E[X]$

and $E[X] = \lim_{n \to \infty} E[X_n]$.

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