1) What is a probability space?

soln: $(\Omega, \mathcal{F}, P)$ is a **probability space** if Ω is a sample space, $\mathcal{F}$ is a σ-field on Ω and P is a probability measure on $(\Omega, \mathcal{F})$.

2) What is a probability measure?

soln: A set function P is a **probability measure** on a $\sigma-$field $\mathcal{F}$ on Ω if P1) $0 \leq P(A) \leq 1$ for $A \in \mathcal{F}$. P2) $P(\emptyset) = 0$ and $P(\Omega) = 1$, P3) If $A_1, A_2, ...$ are disjoint $\mathcal{F}$ sets, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ (countable additivity).

3) What is a random variable?

soln: Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a **random variable** if $X^{-1}(B) \in \mathcal{F} \ \ \forall\ B \in \mathcal{B}(\mathbb{R})$. Equivalently, X is a random variable if $\{X \leq t\} = \{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$.

4) When is a real function a measurable function?

soln: Let $\mathcal{B}(\mathbb{R})$ be the Borel $\sigma-$field on the real numbers $\mathbb{R}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let the real function $X : \Omega \to \mathbb{R}$. Then X is a **measurable function** if $X^{-1}(B) \in \mathcal{F} \ \ \forall\ B \in \mathcal{B}(\mathbb{R})$. Equivalently, X is a measurable function if $\{X \leq t\} = \{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$.

Note: Fix the probability space $(\Omega, \mathcal{F}, P)$. Combining 3) and 4) shows X **is a random variable iff X is a measurable function**.

5) State the Borel Cantelli lemmas and prove the first Borel Cantelli lemma.

soln: Let $(\Omega, \mathcal{F}, P)$ be fixed and A_n events.

1) If $\sum_{n=1}^{\infty} P(A_n) < \infty$ (the sum converges), then $P(limsup_n\ A_n) = 0$.

2) If the A_n are independent events and $\sum_{n=1}^{\infty} P(A_n) = \infty$ (the sum diverges), then $P(limsup_n\ A_n) = 1$.

proof of 1): Since $limsup_n\ A_n \subseteq \bigcup_{k=m}^{\infty} A_k$ for any positive integer m, $P(limsup_n\ A_n) \leq P(\bigcup_{k=m}^{\infty} A_k) \leq \sum_{k=m}^{\infty} P(A_k) \leq \epsilon$ for $m \geq m(\epsilon)$ by defiinition of a convergent sum. Since $\epsilon > 0$ is arbitrary, $P(limsup_n\ A_n) = 0$.

6) State and prove the Monotone Convergence Theorem (for RVs).

7) State and prove the Lebesgue Dominate Convergence Theorem (for RVs).

8) State and prove the Central Limit Theorem.