Solutions Ph.D. Qualifying Examination in Probability
4:00–8:00 Thursday, January 22, 2026

1. a) Let $\boldsymbol{x}_1, ..., \boldsymbol{x}_k$ be independent and identically distributed (iid) with $E(\boldsymbol{x}) = \boldsymbol{\mu}$ where $\boldsymbol{x}$ is $p \times 1$. Let $n = \text{floor(k/2)} = \lfloor \text{k}/2 \rfloor$ be the integer part of $k/2$. So floor(100/2) = floor(101/2) = 50. Let the iid random variables $W_i = \boldsymbol{x}_{2i-1}^T \boldsymbol{x}_{2i}$ for $i = 1, ..., n$. Hence $W_1, W_2, ..., W_n = \boldsymbol{x}_1^T \boldsymbol{x}_2, \boldsymbol{x}_3^T \boldsymbol{x}_4, ..., \boldsymbol{x}_{2n-1}^T \boldsymbol{x}_{2n}$. Then $E(W_i) = \boldsymbol{\mu}^T \boldsymbol{\mu} = \theta$ and $V(W_i) = \sigma_W^2$.

Find the limiting distribution of $\sqrt{n}(\overline{W} - \theta)$.

b) Assume the cases $(\boldsymbol{x}_i^T, Y_i)^T$ are iid. Let $\boldsymbol{\mu}_{\boldsymbol{x}} = E(\boldsymbol{x})$ and $\mu_Y = E(Y)$. Let $\boldsymbol{w}_i = (\boldsymbol{x}_i - \boldsymbol{\mu}_{\boldsymbol{x}})(Y_i - \mu_Y)$ for $i = 1, ..., n$ with sample mean $\overline{\boldsymbol{w}}_n$. Let the $p \times p$ covariance matrix of the $\boldsymbol{w}_i$ be $Cov(\boldsymbol{w}_i) = \boldsymbol{\Sigma}_{\boldsymbol{w}}$. Then $E(\boldsymbol{w}_i) = \boldsymbol{\Sigma}_{\boldsymbol{x},Y} = Cov(\boldsymbol{x}, Y)$.

Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{w}}_n - \boldsymbol{\Sigma}_{\boldsymbol{x},Y})$.

c) Suppose $Y_n \sim U(0, n)$. Then the (cumulative) distribution function of Y_n is $F_n(t) = t/n$ for $0 < t \leq n$ and $F_n(t) = 0$ for $t \leq 0$. Prove or disprove that Y_n converges in distribution (law) to a random variable Y.

Solution: a) $\sqrt{n}(\overline{W} - \theta) \xrightarrow{D} N(0, \sigma_W^2)$ by the CLT.

b) $\sqrt{n}(\overline{\boldsymbol{w}}_n - \boldsymbol{\Sigma}_{\boldsymbol{x},Y}) \xrightarrow{D} N_p(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{w}})$ by the multivariate central limit theorem since the $\boldsymbol{w}_i$ are iid with $E(\boldsymbol{w}_i) = Cov(\boldsymbol{x}, Y)$ and $Cov(\boldsymbol{w}) = \boldsymbol{\Sigma}_{\boldsymbol{w}}$.

c) Note that $\lim_{n\to\infty} F_n(t) = 0$ for $t \leq 0$. If $t > 0$ and $n > t$, then $F_n(t) = t/n \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} F_n(t) = H(t) = 0$ for all t, and Y_n does not converge in distribution to any random variable Y since $H(t) \equiv 0$ is a continuous function but not a cdf.

2. Let $X_1, ..., X_n$ be iid random variables with common moment generating function $E(e^{tX}) = M(t)$. Let $S_n = \sum_{i=1}^n X_i$. Then the moment generating function of S_n is $M_{S_n}(t) = [M(t)]^n$. Note that $e^{tS_n} = e^{t(X_1+\cdots+X_n)} = \prod_{i=1}^n e^{tX_i}$. For any t for which $M(t)$ exists and is finite, let

$$Y_n = \frac{e^{tS_n}}{[M(t)]^n} = \prod_{i=1}^n \frac{e^{tX_i}}{M(t)}.$$

a) Prove that Y_n is a martingale.

b) Does there exist a random variable Y such that $Y_n \xrightarrow{wp1} Y$
(converges with probability one, or almost everywhere, or almost surely to Y)?

Solution: Need to show M1) Y_n is measurable $\mathcal{F}_n$,
M2) $E(|Y_n|) < \infty$,
Then $Y_1, Y_2, ...$ is
M3) a *martingale* if $E(Y_{n+1}|\mathcal{F}_n) = Y_n$ wp1.
To show M2), use $E(|Y_n|) < \infty$ for each n iff $E(Y_n) \in \mathbb{R}$ for each n.
Usually take $\mathcal{F}_n = \sigma(X_1, ..., X_n)$ if Y_n is a function of $X_1, ..., X_n$.
Then M3) becomes $E(Y_{n+1}|\mathcal{F}_n) = E(Y_{n+1}|X_1, ..., X_n) = Y_n$ wp1,

a) 0) Take $\mathcal{F}_n = \sigma(X_1, ..., X_n)$.

M1) holds since Y_n is a (measurable) function of $X_1, ..., X_n$, hence measurable $\mathcal{F}_n$.

M2) holds since

$$E(Y_n) = E(|Y_n|) = E(\prod_{i=1}^{n} W_i) = E\left(\prod_{i=1}^{n} \frac{e^{tX_i}}{M(t)}\right) \stackrel{ind}{=} \prod_{i=1}^{n} \frac{E[e^{tX_i}]}{M(t)} = \prod_{i=1}^{n} \frac{M(t)}{M(t)} = 1.$$

For M3), note that $Y_{n+1} = Y_n \dfrac{e^{tX_{n+1}}}{M(t)}$. Thus

$$E(Y_{n+1}|X_1, ..., X_n) = E\left(Y_n \frac{e^{tX_{n+1}}}{M(t)}|X_1, ..., X_n\right) \stackrel{ind}{=} Y_n E\left(\frac{e^{tX_{n+1}}}{M(t)}\right) = Y_n.$$

b) By M2), $\sup_n E(|Y_n|) = \sup_n E(Y_n) = 1 < \infty$. Hence by the Submartingale Convergence Theorem, there does exist a random variable Y such that $Y_n \stackrel{wp1}{\to} Y$.

3. Suppose $A_1, A_2, ...$ are events such that $\sum_n P(A_n) < \infty$. Find $P(limsup_n\ A_n)$, find $P(liminf_n\ A_n)$, and if $P(A_n) \to c$, find c.

Solution: By the 1st Borel Cantelli Lemma, $P(limsup_n\ A_n) = 0$. thus $P(liminf_n\ A_n) = 0$, and $c = 0$.

4. Fix probability space $(\Omega, \mathcal{F}, P)$. Let $X : \Omega \to \mathbb{R}$. X is a measurable function iff X is a random variable (RV) iff any one of the following conditions holds.
i) $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \ \ \forall\ B \in \mathcal{B}(\mathbb{R})$.
ii) $X^{-1}((-\infty, t]) = \{X \le t\} = \{\omega \in \Omega : X(\omega) \le t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$.
iii) $X^{-1}((-\infty, t)) = \{X < t\} = \{\omega \in \Omega : X(\omega) < t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$.
iv) $X^{-1}([t, \infty)) = \{X \ge t\} = \{\omega \in \Omega : X(\omega) \ge t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$.
v) $X^{-1}((t, \infty)) = \{X > t\} = \{\omega \in \Omega : X(\omega) > t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$.

Let X, Y, and X_i be RVs on $(\Omega, \mathcal{F}, P)$. **Prove the following**.
a) $X + Y$ is a RV.
b) X^2 is a RV.
c) XY is a RV.
d) $inf_n\ X_n$ is a RV.

Solution: a) For each t,

$$\{X + Y < t\} = \cup_{r \in \mathbb{Q}} \ [\{X < r\} \cap \{Y < t - r\}] \in \mathcal{F}$$

since the union is countable. Thus a sum of two random variables is a random variable

b) For any $t \ge 0$, $\{X^2 \le t\} = \{-\sqrt{t} \le X \le \sqrt{t}\} = \{X \le \sqrt{t}\} - \{X < -\sqrt{t}\} \in \mathcal{F}$, while for any $t < 0$, $\{X^2 \le t\} = \emptyset \in \mathcal{F}$. Thus X^2 is a random variable.

c) $XY = 0.5[(X + Y)^2 - X^2 - Y^2]$ is a random variable by b).

d) For each t, $\{\inf_n X_n \ge t\} = \cap_{n=1}^{\infty} \{X_n \ge t\} \in \mathcal{F}$.

5. Suppose $\mathbb{G}$ and $\mathcal{F}$ are σ-fields with $\mathbb{G} \subseteq \mathcal{F}$. Prove that if X is measurable $\mathbb{G}$, then X is measurable $\mathcal{F}$.

Solution: $X^{-1}(B) \in \mathbb{G} \ \ \forall B \in \mathcal{B}(\mathbb{R})$. Thus $X^{-1}(B) \in \mathcal{F} \ \ \forall B \in \mathcal{B}(\mathbb{R})$.

6. State and prove Lebesgue's Dominated Convergence Theorem (for RVs). Ignore "ae" in the proof.

LDCT: If the $|X_n| \le Y$ ae where Y is integrable, and if $X_n \to X$ ae, then X and X_n are integrable and $E(X_n) \to E(X)$.

Proof. Since $limsup\ |X_n| = liminf|X_n| = lim\ |X_n| = |X| \leq Y$, X_n and X are integrable. Using the nonnegativity of $Y - X_n$ and $Y + X_n$,

$$E(Y) - E(X) = E(Y - X) = E[liminf\ (Y - X_n)] \leq$$

$$liminf E(Y - X_n) = E(Y) - limsup\ E(X_n)$$

where the first and third equalities follow by linearity, the second inequality holds since $lim\ (Y - X_n) = liminf\ (Y - X_n) = Y - X$, and the inequality holds by Fatou's lemma. Since $E(Y)$ is finite by integrability,

$$-E(X) \leq -limsup\ E(X_n).$$

Thus $E(X) \geq limsup\ E(X_n)$. Similarly,

$$E(Y) + E(X) = E(Y + X) = E[liminf\ (Y + X_n)] \leq$$

$$liminf E(Y + X_n) = E(Y) + liminf\ E(X_n).$$

Hence

$$E(X) \leq liminf\ E(X_n) \leq limsup\ E(X_n) \leq E(X).$$

Thus $E(X) = \lim_{n\to\infty} E(X_n)$. $\square$

7. Let μ be a measure on $(\Omega, \mathcal{F})$ such that $\mu(\Omega) = d > 0$ is finite. Prove that $P_\mu(A) = \mu(A)/\mu(\Omega)$ for $A \in \mathcal{F}$ is a probability measure on $(\Omega, \mathcal{F})$.

(Recall that a measure has most of the properties of a probability measure, except $\mu(\Omega)$ need not be equal to 1. Hence μ is a nonnegative set function that satisfies countable additivity and $\mu(\emptyset) = 0$. You may use the fact that if $A, B \in \mathcal{F}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.)

Proof. P1) Let $A \in \mathcal{F}$. Then $P_\mu(A) = \mu(A)/\mu(\Omega)$ and $0 \leq P_\mu(A) \leq 1$ since μ is nonnegative and $0 \leq \mu(A) \leq \mu(\Omega)$.

P2) $P_\mu(\Omega) = \mu(\Omega)/\mu(\Omega) = 1$ and $P_\mu(\emptyset) = \mu(\emptyset)/\mu(\Omega) = 0$.

P3) Let $A_1, A_2, ...$ be disjoint $\mathcal{F}$ sets. Then
$P_\mu(\cup_{i=1}^\infty A_i) = \mu(\cup_{i=1}^\infty A_i)/\mu(\Omega) = \sum_{i=1}^\infty [\mu(A_i)/\mu(\Omega)] = \sum_{i=1}^\infty P_\mu(A_i)$.

8. Let $\mathcal{C}_O$ be the class of all open real sets. Let $\mathcal{C}_C$ be the class of all closed real sets. Then the Borel σ-field $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C}_O)$. Prove that $\sigma(\mathcal{C}_O) = \sigma(\mathcal{C}_C)$.

Solution. i) $(\sigma(\mathcal{C}_C) \subseteq \sigma(\mathcal{C}_O))$:
Let set $A \in \mathcal{C}_C$. Then A is a closed set. Thus A^c is open $\Rightarrow A^c \in \sigma(\mathcal{C}_O)$.
Thus $A = (A^c)^c \in \sigma(\mathcal{C}_O)$.
Thus $\mathcal{C}_C \subseteq \sigma(\mathcal{C}_O)$.
Thus $\sigma(\mathcal{C}_C) \subseteq \sigma(\mathcal{C}_O)$.

ii) $(\sigma(\mathcal{C}_O) \subseteq \sigma(\mathcal{C}_C))$:
Let set $O \in \mathcal{C}_O$. Then O^c is closed. Thus $O^c \in \sigma(\mathcal{C}_C)$.
Thus $O = (O^c)^c \in \sigma(\mathcal{C}_C)$.
Thus $\mathcal{C}_O \subseteq \sigma(\mathcal{C}_C)$.
Thus $\sigma(\mathcal{C}_O) \subseteq \sigma(\mathcal{C}_C)$. $\square$

9. Suppose X_n is a sequence of random variables with $P(X_n = 1/n) = 0.5$ and $P(X_n = -1/n) = 0.5$.

a) Show whether or not $X_n \xrightarrow{1} 0$ (convergence in rth mean with $r = 1$).

b) Does $X_n \xrightarrow{D} X$ for some random variable X? Prove or disprove.

Solution. a) $E(|X_n|) = \dfrac{1}{n}1 \to 0$ as $n \to \infty$. Thus $X_n \xrightarrow{1} 0$.

b) $X_n \xrightarrow{D} 0$ since $X_n \xrightarrow{1} 0$.