Ph.D. Qualifying Examination in Probability
4:00–8:00 Thursday, Sept. 3, 2026

1. a) Let $X_1, ..., X_n$ be iid with mean $E(X) = \mu$ and variance $V(X) = \sigma^2 > 0$. Then $n(\overline{X} - \mu)^2 = [\sqrt{n}(\overline{X} - \mu)]^2 \xrightarrow{D} W$. What is W? Hint: use the continuous mapping theorem. Note that $Z \sim N(0, \sigma^2) \sim \sigma N(0, 1)$.

b) Suppose $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ are iid $k \times 1$ random vectors where $E(\boldsymbol{X}_i) = \boldsymbol{1} = (1, ..., 1)^T$ and $Cov(\boldsymbol{X}_i) = \boldsymbol{I}_k = diag(1, ..., 1)$, the $k \times k$ identity matrix. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{X}} - \boldsymbol{c})$ for appropriate vector $\boldsymbol{c}$.

c) Suppose $Y_n \sim U(-n, n)$. Then the (cumulative) distribution function of Y_n is $F_n(t) = \dfrac{t+n}{2n}$ for $-n < t \le n$, $F_n(t) = 0$ for $t \le -n$, and $F_n(t) = 1$ for $t \ge n$. Prove or disprove that Y_n converges in distribution (law) to a random variable Y.

Solution. a) Since $\sqrt{n}(\overline{X} - \mu) \xrightarrow{D} Z \sim N(0, \sigma^2) \sim \sigma N(0, 1)$ by the CLT, $W = Z^2 \sim \sigma^2 \chi^2_1$.

b) $\sqrt{n}(\overline{\boldsymbol{X}} - \boldsymbol{1}) \xrightarrow{D} N_k(\boldsymbol{0}, \boldsymbol{I}_k)$

c) $\lim_{n\to\infty} F_n(t) = H(t) = 0.5$ for all $t \in \mathbb{R}$. $H(t)$ is continuous but not a cdf. Thus X_n does not converge in distribution to any RV X.

2. Let $X_1, X_2, ...$ be independent nonnegative random variables with $E(X_k) = 1$ for $k = 1, 2, ...$ Let $Y_n = \prod_{i=1}^n X_i$. i) Show that $\{Y_n\}$ is a martingale.
ii) Is there a random variable Y such that $Y_n \xrightarrow{wp1} Y$? Explain.

Solution. i) Need to show M1) Y_n is measurable $\mathcal{F}_n$, M2) $E(|Y_n|) < \infty$. Then $Y_1, Y_2, ...$ is M3) a *martingale* if $E(Y_{n+1}|\mathcal{F}_n) = Y_n$ wp1.
Usually take $\mathcal{F}_n = \sigma(X_1, ..., X_n)$ if Y_n is a function of $X_1, ..., X_n$. Then M3) becomes $E(Y_{n+1}|\mathcal{F}_n) = E(Y_{n+1}|X_1, ..., X_n) = Y_n$ wp1.

0) Take $\mathcal{F}_n = \sigma(X_1, ..., X_n)$.

M1) holds since the product Y_n is a RV and a function of $X_1, ..., X_n$ (hence measurable $\mathcal{F}_n$).

M2) holds since $E(Y_n) = E(|Y_n|) = \prod_{i=1}^n E(X_i) = 1 < \infty$ for $n = 1, 2, ...$

M3) holds since $E(Y_{n+1}|X_1, ..., X_n) = E(Y_n X_{n+1}|X_1, ..., X_n) = Y_n E(X_{n+1}|X_1, ..., X_n) = Y_n E(X_{n+1}) = Y_n$ for each n.

ii) Since the X_i are nonnegative, $E(|Y_n|) = E(Y_n) = \prod_{i=1}^n E(X_i) = 1 = K$. Thus by the Submartingale Convergence Theorem, there does exist RV Y such that $Y_n \xrightarrow{wp1} Y$.

3. Suppose $A_1, A_2, ...$ are events such that $\sum_n P(A_n^c) < \infty$. Find $P(limsup_n\ A_n)$, find $P(liminf_n\ A_n)$, and if $P(A_n) \to c$, find c.

Solution. By the 1st Borel Cantelli Lemma, $P(limsup_n\ A_n^c) = 0 = P[(liminf_n\ A_n)^c]$. So $P(liminf_n\ A_n) = 1 = P(limsup_n\ A_n)$, and $c = 1$.

4. Fix probability space $(\Omega, \mathcal{F}, P)$. Let $X : \Omega \to \mathbb{R}$. X is a measurable function iff X is a random variable (RV) iff any one of the following conditions holds.
i) $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \ \ \forall \ B \in \mathcal{B}(\mathbb{R})$.
ii) $X^{-1}((-\infty, t]) = \{X \leq t\} = \{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{F} \ \forall t \in \mathbb{R}$.
iii) $X^{-1}((-\infty, t)) = \{X < t\} = \{\omega \in \Omega : X(\omega) < t\} \in \mathcal{F} \ \forall t \in \mathbb{R}$.
iv) $X^{-1}([t, \infty)) = \{X \geq t\} = \{\omega \in \Omega : X(\omega) \geq t\} \in \mathcal{F} \ \forall t \in \mathbb{R}$.
a) Prove that X is a RV iff $X^{-1}((t, \infty)) = \{X > t\} = \{\omega \in \Omega : X(\omega) > t\} \in \mathcal{F} \ \forall t \in \mathbb{R}$.

Let X, Y, and X_i be RVs on $(\Omega, \mathcal{F}, P)$.
b) Suppose $limsup_n \ X_n$ and $liminf_n \ X_n$ are RVs. If $lim_n X_n = X$, prove that X is a RV.

Solution. a) For each t, $\{X \leq t\} \in \mathcal{F}$ iff $\{X \leq t\}^c = \{X > t\} \in \mathcal{F}$.
b) $X = limsup_n \ X_n = liminf_n \ X_n$ is a RV.

5. Suppose Λ is the index set for $\sigma-$fields on Ω, $\mathcal{F}_\lambda$, that contain a class $\mathcal{A}$ of subsets of Ω. Then Λ is nonempty since the $\sigma-$field of all subsets of Ω contains $\mathcal{A}$. Let the $\sigma-$field generated by $\mathcal{A}$ be

$$\sigma(\mathcal{A}) = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda.$$

Prove that $\sigma(\mathcal{A})$ is a $\sigma-$field.

Solution. i) $\Omega \in \sigma(\mathcal{A})$ since $\Omega \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$.

ii) If $A \in \sigma(\mathcal{A})$, then $A \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$. Hence $A^c \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$. Thus $A^c \in \sigma(\mathcal{A})$.

iii) If $A_1, A_2, ... \in \sigma(\mathcal{A})$, then $A_1, A_2, ... \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$. Hence $\cup_{i=1}^\infty A_i \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$. Thus $\cup_{i=1}^\infty A_i \in \sigma(\mathcal{A})$.

6. Prove the following theorem. In your proof of iii) and iv), you may use ii) **linearity**: If X and Y are integrable and $a, b \in \mathbb{R}$, then $aX + bY$ is integrable with $E(aX + bY) = aE(X) + bE(Y)$.

Theorem. i) X is integrable iff both $E[X^+]$ and $E[X^-]$ are finite.
iii) **monotonicity**: If X and Y are integrable and $X \leq Y$ ae, then
$E(X) \leq E(Y)$.
iv) $|E(X)| \leq E(\ |X| \ )$.

Solution. i) If X is integrable, then $E[|X|] = E[X^+] + E[X^-]$. Since $E[X^+] \geq 0$, $E[X^-] \geq 0$, and the sum is finite, both terms are finite. If both $E[X^+]$ and $E[X^-]$ are finite, then $E[|X|] = E[X^+] + E[X^-]$ is finite.

iii) By ii) $0 \leq E(Y - X) = E(Y) - E(X)$. Thus $E(Y) \geq E(X)$.

iv) Since $-|X| \leq X \leq |X|$, iii) implies that $E(X) \leq E(|X|)$ and $-E(|X|) \leq E(X)$. Thus $-E(X) \leq E(|X|)$. Hence $|E(X)| \leq E(\ |X| \ )$.

7. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Thus $\mathcal{F}$ is a σ-field, and P is a probability measure on $(\Omega, \mathcal{F})$. Let $A, B \in \mathcal{F}$ satisfy $0 < P(A) < 1$ and $0 < P(B) < 1$. Let A^c be the complement of A.

a) Prove that $P(A^c) = 1 - P(A)$.

b) Suppose that $P(A \cap B) = P(A)P(B)$. Show that $A \cap B \neq \emptyset$.

c) Show that if $D \in \mathcal{F}$ and $P(D) = 0$, then $P(A \cap D) = P(A)P(D)$ for every $A \in \mathcal{F}$.

d) Show that if $C \in \mathcal{F}$ and $P(C) = 1$, then $P(A \cap C) = P(A)P(C)$ for every $A \in \mathcal{F}$. Hint: Split A by C, C^c. So $A = A \cap (C \cup C^c) = A \cap \Omega$.

e) Suppose $P(A \cap B) = P(A)P(B)$. Show that $P(A \cap B^c) = P(A)P(B^c)$. Hint: $A = A \cap (B \cup B^c) = A \cap \Omega$.

Solution. a) $A \cup A^c = \Omega$. Thus $P(A \cup A^c) = P(A) + P(A^c) = P(\Omega) = 1$. So $P(A^c) = 1 - P(A)$.

b) $P(\emptyset) = 0 < P(A)P(B)$.

c) $0 \leq P(A \cap D) \leq P(D) = 0 = P(A)P(D)$.

d) $P(A) = P(A \cap \Omega) = P[A \cap (C \cup C^c)] = P[(A \cap C) \cup (A \cap C^c)] = P(A \cap C) + P(A \cap C^c) = P(A \cap C)$ by c) since $P(C^c) = 0$. Thus $P(A) = P(A)P(C) = P(A \cap C)$.

e) $P(A) = P(A \cap \Omega) = P[A \cap (B \cup B^c)] = P[(A \cap B^c) \cup (A \cap B)] = P(A \cap B^c) + P[(A \cap B) = P(A \cap B^c) + P(A)P(B)$. Thus $P(A \cap B^c) = P(A) - P(A)P(B) = P(A)[1 - P(B)] = P(A)P(B^c)$ by a).

8. Let $\mathcal{C}_O$ be the class of all open real sets. Then the Borel σ-field $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C}_O)$. Let $\mathcal{C}_{OI}$ be the class of all open intervals in $\mathbb{R} = (-\infty, \infty)$. Prove that $\sigma(\mathcal{C}_O) = \sigma(\mathcal{C}_{OI})$. Here $\sigma(\mathcal{A})$ is the σ-field generated by $\mathcal{A}$: the smallest σ-field containing $\mathcal{A}$.

Solution. i) $(\sigma(\mathcal{C}_O) \subseteq \sigma(\mathcal{C}_{OI}))$:
Let $O \in \mathcal{C}_O$. Then $O = \cup_{i=1}^{\infty} (a_i, b_i)$.
Thus $O \in \sigma(\mathcal{C}_{OI})$.
Thus $\mathcal{C}_O \subseteq \sigma(\mathcal{C}_{OI})$.
Thus $\sigma(\mathcal{C}_O) \subseteq \sigma(\mathcal{C}_{OI})$.

ii) $(\sigma(\mathcal{C}_{OI}) \subseteq \sigma(\mathcal{C}_O))$:
$\mathcal{C}_{OI} \subseteq \mathcal{C}_O$.
Thus $\mathcal{C}_{OI} \subseteq \sigma(\mathcal{C}_O)$.
Thus $\sigma(\mathcal{C}_{OI}) \subseteq \sigma(\mathcal{C}_O)$.

9. Suppose X_n is a sequence of random variables with $P(X_n = -n) = 1/n$ and $P(X_n = 1/n) = 1 - 1/n$.

a) Show whether or not $X_n \xrightarrow{1} 0$ (convergence in rth mean with $r = 1$).

b) Does $X_n \xrightarrow{D} X$ for some random variable X? Prove or disprove.

Solution. a) $E|X_n| = \dfrac{1}{n}\left(1 - \dfrac{1}{n}\right) + n\dfrac{1}{n} = \dfrac{1}{n} - \dfrac{1}{n^2} + 1 \to 1 \neq 0$ as $n \to \infty$. Thus X_n does not converge in 1th mean to 0.

b) $F_n(x) \to 0$ for $x \leq 0$, and $F_n(x) \to 1$ for $x > 0$ which is the cdf of X = the point mass at 0 except at $x = 0$. Hence $X_n \xrightarrow{D} X$ where $P(X = 0) = 1$.