The oral exam problems are useful short qual problems or parts of qual problems.

1) State and prove the First Borel Cantelli Lemma.

Let $(\Omega, \mathcal{F}, P)$ be fixed and A_n events.
If $\sum_{n=1}^{\infty} P(A_n) < \infty$ (the sum converges), then $P(limsup_n\ A_n) = 0$.

proof: Since $limsup_n\ A_n \subseteq \bigcup_{k=m}^{\infty} A_k$ for any positive integer m, $P(limsup_n\ A_n) \leq P(\bigcup_{k=m}^{\infty} A_k) \leq \sum_{k=m}^{\infty} P(A_k) \leq \epsilon$ for $m \geq m(\epsilon)$ by defiinition of a convergent sum. Since $\epsilon > 0$ is arbitrary, $P(limsup_n\ A_n) = 0$.

2) State and prove the Second Borel Cantelli Lemma.

If the A_n are independent events and $\sum_{n=1}^{\infty} P(A_n) = \infty$ (the sum diverges), then $P(limsup_n\ A_n) = 1$.

proof:

3) State and prove the Monotone Convergence Theorem (for RVs).

4) State and prove the Lebesgue Dominate Convergence Theorem (for RVs).

5) State and prove the Central Limit Theorem.