Math 581 Exam 1 is Thursday, Sept. 16, 2:00-3:15 NO NOTES. CHECK FORMULAS: YOU ARE RESPONSIBLE FOR ANY ERRORS ON THIS HANDOUT!

1) The **sample space** Ω is the set of all outcomes from an idealized experiment. The **empty set** is $\emptyset$. The **complement of a set** A is $A^c = \{\omega \in \Omega : \omega \notin A\}$.

2) Let $\Omega \neq \emptyset$. A class $\mathcal{F}$ of subsets of Ω is a $\sigma-$field (or $\sigma-$algebra) on Ω if
i) $\Omega \in \mathcal{F}$.
ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.
iii) $A_1, A_2, ... \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.
Note that i), ii), and iii) mean that a σ-field is a field (or algebra) on Ω. A $\sigma-$field is closed under countable set operations. The term "on Ω" is often understood and omitted.

Common error: Use n instead of ∞ in iv).

3) De Morgan's laws: i) $A \cap B = (A^c \cup B^c)^c$, ii) $A \cup B = (A^c \cap B^c)^c$,

$$iii) \; [\cup_{i=1}^{\infty} A_i]^c = \cap_{i=1}^{\infty} A_i^c.$$

4) Let $\mathcal{A}$ be a class of sets. The $\sigma-$field generated by $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$ is the intersection of all $\sigma-$fields containing $\mathcal{A}$. Then $\sigma(\mathcal{A})$ is the smallest $\sigma-$field containing $\mathcal{A}$.

5) Let $\mathcal{A}$ be the class of all open intervals of [0,1]. Then $\sigma(\mathcal{A}) = \mathcal{B}[0,1]$ is the Borel $\sigma-$field on [0,1]. Fact: $\mathcal{B}[0,1] = \sigma(\mathcal{A})$ where $\mathcal{A}$ is the class of all closed intervals in [0,1], or $\mathcal{A}$ is the class of all intervals of the form $(a, b]$ in [0,1], or $\mathcal{A}$ is the class of all intervals of the form $[a, b)$ in [0,1].

6) A set function P is a **probability measure** on a $\sigma-$field $\mathcal{F}$ on Ω if P1) $0 \leq P(A) \leq 1$ for $A \in \mathcal{F}$. P2) $P(\emptyset) = 0$ and $P(\Omega) = 1$, P3) If $A_1, A_2, ...$ are disjoint $\mathcal{F}$ sets, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ (countable additivity).

Common error: use n instead of ∞ in P3).

7) $A - B = A \cap B^c$ is the difference between A and B.

8) $A_n \uparrow A$ means $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \cup_{i=1}^{\infty} A_i$.
$A_n \downarrow A$ means $A_1 \supseteq A_2 \supseteq \cdots$ and $A = \cap_{i=1}^{\infty} A_i$.
$x_n \uparrow x$ means $x_1 \leq x_2 \leq \cdots$ and $x_n \to x$.
$x_n \downarrow x$ means $x_1 \geq x_2 \geq \cdots$ and $x_n \to x$.

9) Properties of P: Let A, B, A_i, A_n, A_k be $\mathcal{F}$ sets.
i) Finite additivity: If $A_1, ..., A_n$ are disjoint, then $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$.
ii) P is monotone: $A \subseteq B \Rightarrow P(A) \leq P(B)$.
iii) If $A \subseteq B$, then $P(B - A) = P(B) - P(A)$.
iv) Complement rule: $P(A^c) = 1 - P(A)$.
v) Finite subadditivity: $P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$.
vi) continuity from below: If $A_n \uparrow A$ then $P(A_n) \uparrow P(A)$.
vii) continuity from above: If $A_n \downarrow A$ then $P(A_n) \downarrow P(A)$.
viii) countable subadditivity: $P(\cup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} P(A_k)$.

Note: vi) and vii) together are known as monotone continuity.

10) $\overline{lim}\ A_n = limsup_n\ A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \{\omega : \omega \in A_n$ for infinitely many $A_n\}$.
$\underline{lim}\ A_n = liminf_n\ A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = \{\omega : \omega \in A_n$ for all but finitely many $A_n\}$.
If $A_n \in \mathcal{F}$, then $\overline{lim}\ A_n, \underline{lim}\ A_n \in \mathcal{F}$. Also, $liminf_n\ A_n \subseteq limsup_n\ A_n$.

11) If $liminf_n\ A_n = limsup_n\ A_n$, then $lim_n\ A_n = A = liminf_n\ A_n = limsup_n\ A_n$, written $A_n \to A$.

If $A_n \in \mathcal{F}$, then $lim_n\ A_n = A \in \mathcal{F}$.

Facts: $(limsup_n\ A_n)^c = liminf_n\ A_n^c$ and $(liminf_n\ A_n)^c = limsup_n\ A_n^c$

12) $(\Omega, \mathcal{F}, P)$ is a **probability space** if Ω is a sample space, $\mathcal{F}$ is a σ-field on Ω and P is a probability measure on $(\Omega, \mathcal{F})$. Then an **event** A is any set $A \in \mathcal{F}$.

13) For a sequence of real numbers, $\overline{lim}\ x_n = limsup_n\ x_n = inf_n\ sup_{k \geq n}\ x_k$, and $\underline{lim}\ x_n = liminf_n\ x_n = sup_n\ inf_{k \geq n}\ x_k$. Also, $\overline{lim}\ (-x_n) = -\underline{lim}\ x_n$

inf=infinum = greatest lower bound, sup = supremum = least upper bound

Fact 1) $\underline{lim}\ x_n \leq \overline{lim}\ x_n$. Fact 2) $lim_n x_n = x$ iff $x = \underline{lim}\ x_n = \overline{lim}\ x_n$. Then $x_n \to x$. Fact 3) If $\{x_n\}$ is a bounded sequence, then $\overline{lim}\ x_n$ = largest accumulation point (cluster point) of $\{x_n\}$, and $\underline{lim}\ x_n$ = smallest accumulation point of $\{x_n\}$.

14) Theorem 4.1: For each sequence $\{A_n\}$ of $\mathcal{F}$ sets,
i) $P(liminf_n\ A_n) \leq liminf_n\ P(A_n) \leq limsup_n\ P(A_n) \leq P(limsup_n\ A_n)$
ii) Continuity of probability: If $A_n \to A$, then $P(A_n) \to P(A)$.

15) Let $A_1, A_2, ...$ be $\mathcal{F}$ sets.
i) If $P(A_i) = 0$ for all i, then $P(\cup_{i=1}^{\infty} A_i) = 0$.
ii) If $P(A_i) = 1$ for all i, then $P(\cap_{i=1}^{\infty} A_i) = 1$.

16) i) Two events A and B are **independent**, written $A \perp\!\!\!\perp B$, if $P(A \cap B) = P(A)P(B)$.

ii) A finite collection of events $A_1, ..., A_n$ is **independent** if for *any* subcollection $A_{i_1}, ..., A_{i_k}$, $P(\cap_{j=1}^{k} A_{i_j}) = \prod_{j=1}^{k} P(A_{i_j})$.

iii) An infinite (perhaps uncountable) collection of events is **independent** if each of its finite subcollections is.

If the events are not independent, then the events are dependent.

17) **Borel-Cantelli Lemmas**: Let $(\Omega, \mathcal{F}, P)$ be fixed and A_n events.
1) If $\sum_{n=1}^{\infty} P(A_n) < \infty$ (the sum converges), then $P(limsup_n\ A_n) = 0$.
2) If the A_n are independent events and $\sum_{n=1}^{\infty} P(A_n) = \infty$ (the sum diverges), then $P(limsup_n\ A_n) = 1$.

18) Let $\{A_n\}$ be a sequence of events defined on $(\Omega, \mathcal{F}, P)$. Then $\tau = \cap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, ...)$ is the **tail $\sigma-$field**. (See 4) on the exam 1 review.) If $A \in \tau$, then A is a **tail event**.

19) The Kolmogorov 0-1 Law: Let $\{A_n\}$ be a sequence of independent events defined on $(\Omega, \mathcal{F}, P)$. If $A \in \tau$, then $P(A) = 0$ or $P(A) = 1$.

20) Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a **random variable** if $X^{-1}(B) \in \mathcal{F}\ \ \forall\ B \in \mathcal{B}(\mathbb{R})$. Equivalently, X is a random variable if
$\{X \leq t\} = \{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{F}\ \forall t \in \mathbb{R}$.

21) The random variable X is a **measurable function**. $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma-$field on the real numbers $\mathbb{R} = (-\infty, \infty)$. The **inverse image** $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$. Note that the inverse image $X^{-1}(B)$ is a set. $X^{-1}(B)$ is **not the inverse function**.

22) For now, let the expected value $E(X)$ and the variance $V(X) = VAR(X) = E[(X - E(X))^2] = E(X^2) - [E(X)]^2$ be as in a calculus based probability course.

23) **Generalized Chebyshev's Inequality** = *Generalized Markov's Inequality*: Let $u : \mathbb{R} \to [0, \infty)$ be a nonnegative function. If $E[u(Y)]$ exists then for any $c > 0$,

$$P[u(Y) \geq c] \leq \frac{E[u(Y)]}{c}.$$

If $\mu = E(Y)$ exists, then taking $u(y) = |y - \mu|^r$ and $\tilde{c} = c^r$ gives
Markov's Inequality: for $r > 0$ and any $c > 0$,

$$P(|Y - \mu| \geq c] = P(|Y - \mu|^r \geq c^r] \leq \frac{E[|Y - \mu|^r]}{c^r}.$$

If $r = 2$ and $\sigma^2 = VAR(Y)$ exists, then we obtain
Chebyshev's Inequality:

$$P(|Y - \mu| \geq c] \leq \frac{VAR(Y)}{c^2}.$$

24) i) The **moment generating function** (mgf) of a random variable X is $M(t) = E[e^{tX}]$ if the expectation exists for t in some neighborhood of 0. Otherwise, the mgf does not exist.
ii) If the mgf $M(t)$ exists, then the **cumulant generating function** (cgf) of a random variable Y is $C(t) = \log M(t)$ where $\log = \log_e = \ln$.

25) Let $g^{(k)}(t)$ be the kth derivative of g with $g' = g^{(1)}$ and $g'' = g^{(2)}$.
i) $E[X^k] = M^{(k)}(0)$, the kth derivative of $M(t)$ evaluated at 0, for positive integers k.
ii) $C'(0) = E(X)$ and $C''(0) = V(X)$.

26) $(\Omega, \mathcal{F})$ is a **measurable space** if Ω is a sample space and $\mathcal{F}$ is a $\sigma-$field on Ω.

27) Let $\mathcal{B}(\mathbb{R})$ be the Borel $\sigma-$field on the real numbers $\mathbb{R} = (-\infty, \infty)$. Let $(\Omega, \mathcal{F})$ be a measurable space, and let the real function $X : \Omega \to \mathbb{R}$. Then X is a **measurable function** if $X^{-1}(B) \in \mathcal{F} \;\; \forall \; B \in \mathcal{B}(\mathbb{R})$. Equivalently, X is a measurable function if $\{X \leq t\} = \{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{F} \; \forall t \in \mathbb{R}$.

28) Fix the probability space $(\Omega, \mathcal{F}, P)$. Combining 20) and 27) shows X **is a random variable iff** X **is a measurable function**.

29) A set function μ is a **measure** on $(\Omega, \mathcal{F})$ (where $\mathcal{F}$ is a $\sigma-$field on Ω) if
m1) $\mu(A) \in [0, \infty]$ for $A \in \mathcal{F}$. (Note that ∞ is allowed.)
m2) $\mu(\emptyset) = 0$, and
m3) If $A_1, A_2, ...$ are disjoint $\mathcal{F}$ sets, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ (countable additivity).

30) A measure μ is **finite** if $\mu(\Omega) < \infty$ and **infinite** if $\mu(\Omega) = \infty$. If $\Omega = \cup_{i=1}^{\infty} A_i$ where $A_i \in \mathcal{F}$ with $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$, then μ is $\sigma-$finite. A measure is a probability measure if $\mu(\Omega) = 1$, and every probability measure is a finite measure and a $\sigma-$finite measure.

31) $(\Omega, \mathcal{F}, \mu)$ is a **measure space** if Ω is a sample space, $\mathcal{F}$ is a $\sigma-$field on Ω, and μ is a measure on $(\Omega, \mathcal{F})$.

32) Properties of a measure μ: Let A, B, A_i, A_n, A_k be $\mathcal{F}$ sets.
i) μ is monotone: $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.
ii) If $A \subseteq B$ and $\mu(B) < \infty$, then $\mu(B - A) = \mu(B) - \mu(A)$.
iii) Finite additivity: If $A_1, ..., A_n$ are disjoint, then $\mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$.
iv) Finite subadditivity: $\mu(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \mu(A_i)$.
v) continuity from above: If $A_n \downarrow A$ and $\mu(A_1) < \infty$, then $\mu(A_n) \downarrow \mu(A)$.
vi) continuity from below: If $A_n \uparrow A$ then $\mu(A_n) \uparrow \mu(A)$.
vii) countable subadditivity: $\mu(\cup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$.

33) Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces. For a mapping $T : \Omega \to \Omega'$, the mapping T is **measurable** $\mathcal{F}/\mathcal{F}'$ if $T^{-1}(A') \in \mathcal{F}$ for each $A' \subseteq \mathcal{F}'$. If $\Omega' = \mathbb{R}^k$ and

$\mathcal{F}' = \mathcal{B}(\mathbb{R}^k)$, and $\boldsymbol{X} : \Omega \to \mathbb{R}^k$, then $\boldsymbol{X}$ is a **measurable function** if $\boldsymbol{X}$ is measurable $\mathcal{F}/\mathcal{B}(\mathbb{R}^k)$ iff $X = \boldsymbol{X}$ is a random variable for $k = 1$ and $\boldsymbol{X}$ is a $1 \times k$ random vector for $k > 1$ iff $\boldsymbol{X}^{-1}(B) = \{\omega : \boldsymbol{X}(\omega) \in B\} \in \mathcal{F}\ \forall B \in \mathcal{B}(\mathbb{R}^k)$.

Note the random vector $\boldsymbol{X} = (X_1, ..., X_k)$ and $\boldsymbol{X}(\omega) = (X_1(\omega), ..., X_k(\omega))$ where the $X_i : \Omega \to \mathbb{R}$ are random variables (measurable functions) for $i = 1, ..., k$.

34) An *indicator* I_A is the function such that $I_A(\omega) = 1$ if $\omega \in A$ and $I_A(\omega) = 0$ if $\omega \notin A$.

35) A function f is a *simple function* if $f = \sum_{i=1}^k x_i I_{A_i}$ for some positive integer k. Thus a simple function f has finite range.

36) A simple function is a random variable if each $A_i \in \mathcal{F}$.