Math 581 — Probability and Measure
Prereq Lebesgue Measure and Integration
\$ 112

p16 1) The *sample space* Ω (Omega) is the set of all possible outcomes from an idealized experiment.

ex) Flip coin once $\Omega = \{\text{heads}, \text{tails}\}$

(idealized since outcomes $\omega \notin \Omega$, such as landing on an edge against a wall, are not allowed)

An element $\omega \in \Omega$ (Omega) is a sample point.

p17 2) Def Let $\Omega \neq \phi$. A class $\mathcal{a}$ of subsets of Ω is a *field* on Ω = algebra on Ω
if
i) $\Omega \in \mathcal{a}$
ii) $A \in \mathcal{a} \Rightarrow A^c \in \mathcal{a}$
iii) $A, B \in \mathcal{a} \Rightarrow A \cup B \in \mathcal{a}$

A^c is the complement of $A = \{\omega \in \Omega : \omega \notin A\}$
$[A^c]^c = A$, the empty set $\phi = \Omega^c$

iii) can be replaced by iii)' $A, B \in \mathcal{a} \Rightarrow A \cap B \in \mathcal{a}$
or by iii)'' $A_1, \ldots, A_n \in \mathcal{a} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{a}$.
A field contains Ω and is closed

under the formation of complements and is finite unions and intersections.

3) De Morgan's laws

$A \cap B = (A^c \cup B^c)^c$

$A \cup B = (A^c \cap B^c)^c$

p17 4) Def] Know Let $\Omega \neq \phi$, A class $\mathcal{F}$ of subsets of Ω is a σ-field on Ω = σ-algebra on Ω if

i) $\Omega \in \mathcal{F}$

ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

iii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

iv) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow$

$$A_1 \cup A_2 \cup \cdots = \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

Common error: Student uses n instead of ∞

A σ-field is a field that is closed under complementation and countable unions and intersections.

p19 5] $\left[\bigcup_{i=1}^{\infty} A_i\right]^c = \bigcap_{i=1}^{\infty} A_i^c$

6] The largest σ-field consists of all subsets of Ω. The smallest σ-field is $\mathcal{F} = \{\phi, \Omega\}$.

Examples 2.2 — 2.4 are good.

7] Know Def] Let $\mathcal{A}$ be a class of sets. The σ-field generated by $\mathcal{A}$, $\sigma(\mathcal{A})$, is the intersection of all σ-fields containing $\mathcal{A}$.

$\sigma(\mathcal{A})$ is the smallest σ-field containing $\mathcal{A}$ since if $\mathcal{F}_i$ is a σ-field containing $\mathcal{A}$, then $\sigma(\mathcal{A}) \subseteq \mathcal{F}_i$

ex) Let $\mathcal{A}$ be the class of all open intervals in $[0,1] = \Omega$. Then

$\sigma(\mathcal{A}) = \mathcal{B}[0,1]$ is the Borel σ-field on $[0,1]$.

$\mathcal{B}[0,1] = \sigma(\mathcal{A})$ where
$\mathcal{A}$ = all closed intervals in $[0,1]$
$\mathcal{A}$ = all intervals of the form $(a,b]$ in $[0,1]$ "
" $[a,b)$

Know
Prob 8] on Ω Def) A set function P on a σ-field $\mathcal{F}$ is a probability measure if
P1) $0 \le P(A) \le 1$ for $A \in \mathcal{F}$
P2) $P(\emptyset) = 0$ and $P(\Omega) = 1$
P3) If $A_1, A_2, \ldots$ are disjoint $\mathcal{F}$ sets then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

(countable additivity). Common error: use $\cap$ instead of $\cup$

Note: $A_1, A_2, \ldots$ are disjoint or mutually exclusive if $A_i \cap A_j = \phi$ for $i \neq j$.
A_i and $B = \phi$ are disjoint for any $A_i \in \mathcal{F}$.

Fact: If $A_1, \ldots, A_n \in \mathcal{F}$ are disjoint, then $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

(finite additivity).

proof: Let $A_i = \phi$ for $i = n+1, n+2, \ldots$.
Then $A_1, A_2, \ldots$ are disjoint.
Thus $P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

$= \sum_{i=1}^{n} P(A_i)$.

9] know $(\Omega, \mathcal{F}, P)$ is a probability space = probability measure space if Ω is a sample space, $\mathcal{F}$ a σ-field in Ω and P is a probability measure.

10] For a discrete random variable RV X, Ω is a countable set and $\mathcal{F}$ is the σ-field of all subsets of Ω.
For a continuous RV X, $\mathcal{F}$ is often the Borel σ-field $\mathcal{B}(\Omega)$ where Ω is an interval.

ex] Let μ_L be the Lebesgue measure on $\Omega = [a,b]$: $\mu_L([c,d]) = d-c$ if $[c,d] \subseteq [a,b]$. The uniform RV has $P = \frac{\mu_L}{\mu_L[a,b]} = \frac{\mu_L}{b-a}$.

$[a,b] = [0,1]$ is interesting. See §1.3.

11] $A - B = A \cap B^c$ = difference between A and B = A minus B

$A \Delta B = (A-B) \cup (B-A)$ = symmetric difference between A and B

p22 12] Properties i) P is monotone:

$A, B \in \mathcal{F}$ and $A \subseteq B \Rightarrow P(A) \leq P(B)$

ii) $P(B-A) = P(B) - P(A)$ if $A \subseteq B$

iii) $P(A^c) = 1 - P(A)$ complement rule

iv) $A_i \in \mathcal{F} \Rightarrow$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

(finite subadditivity, Boole's inequality)

Proof] i), ii) If $A \subseteq B$, then

$B = \underbrace{A \cup (B-A)}_{\text{disjoint}}$.

Thus $P(B) = P(A) + P(B-A) \geq P(A)$

iii) Take $B = \Omega = \underbrace{A \cup A^c}_{\text{disjoint}}$

so $P(\Omega) = 1 = P(A) + P(A^c)$

3.5

iii) Let $B_1 = A_1$ and

$$B_k = A_k \cap A_1^c \cap \cdots \cap A_{k-1}^c = A_k \cap \left[\bigcup_{i=1}^{k-1} A_i\right]^c$$

claim: The B_k are disjoint:
WLOG let $j<k$. Then
$B_j \subseteq A_j$ and $B_k \subseteq A_j^c$.

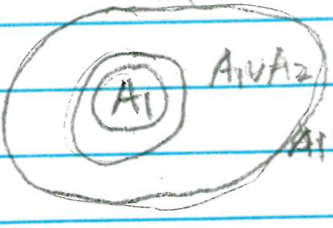
$$\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k \text{ since}$$

$$\bigcup_{k=1}^{n} A_k = A_1 \cup [A_2 \cap A_1^c] \cup [A_3 \cap (A_1 \cup A_2)^c] \cup \cdots \cup [A_n \cap (A_1 \cup \cdots \cup A_{n-1})^c] = \bigcup_{k=1}^{n} B_k$$

for each $n \geq 1$.

$$\text{So } P\left(\bigcup_{k=1}^{n} A_k\right) = P\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{k=1}^{n} P(B_k)$$

$$\underset{\uparrow B_k \subseteq A_k}{\leq} \sum_{k=1}^{n} P(A_k).$$



etc or use induction

13] $A_n \uparrow A$ means $A_1 \subseteq A_2 \subseteq \dots$
and $A = \bigcup_{n=1}^{\infty} A_n$

$A_n \downarrow A$ means $A_1 \supseteq A_2 \supseteq \dots$,
and
$$A = \bigcap_{n=1}^{\infty} A_n.$$

$x_n \uparrow x$ means $x_1 \le x_2 \le \dots$ and $x_n \to x$
$x_n \downarrow x$ means $x_1 \ge x_2 \ge \dots$ and $x_n \to x$.

p23 14] more properties i) and ii) are monotone continuity

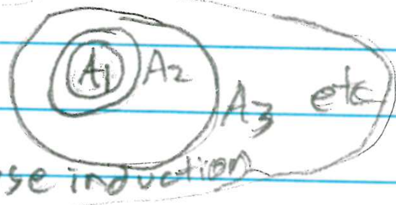
i) continuity from below: If $A_n, A \in \mathcal{F}$ and $A_n \uparrow A$, then $P(A_n) \uparrow P(A)$.

ii) Continuity from above: If $A_n, A \in \mathcal{F}$ and $A_n \downarrow A$, then $P(A_n) \downarrow P(A)$.

in $\mathcal{F}$ if $A_n \in \mathcal{F}$

iii) countable subadditivity: If A_k and $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$, then
$$P\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} P(A_k).$$

proof] i) Let $B_1 = A_1$ and $B_k = A_k - A_{k-1}$.
The B_k are disjoint, since $A_n \uparrow A$,



$$A_n = \bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k \text{ and}$$

or use induction

$$A = \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k.$$ see HW1 #1.

($w \in A \Rightarrow w \in A_i$ for some i. So $w \in A_i = \bigcup_{k=1}^{i} B_k \subseteq \bigcup_{k=1}^{\infty} B_k$

so $A \subseteq \bigcup_{k=1}^{\infty} B_k$. Now $w \in \bigcup_{k=1}^{\infty} B_k$

means $w \in B_j$ for some j. so

$w \in \bigcup_{k=1}^{j} B_k = A_j \subseteq \bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} A_k = A$.

So $P(A) = \sum_{k=1}^{\infty} P(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$

$= \lim_{n\to\infty} P(A_n)$.

ii) $A_n \downarrow A \Rightarrow A_n^c \uparrow A^c$ so

$P(A_n^c) = [1 - P(A_n)] \uparrow [1 - P(A)] = P(A^c)$.

Thus $P(A_n) \downarrow P(A)$.

iii) Let $B_n = \bigcup_{k=1}^{n} A_k$.

$P(B_n) = P\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} P(A_k) \leq \sum_{k=1}^{\infty} P(A_k)$

for any n.

$B_n \uparrow B = \bigcup_{k=1}^{\infty} A_k$ so by i)

$P(B_n) \uparrow P(B) = P\left(\bigcup_{k=1}^{\infty} A_k\right)$

$P(B)$ is the least upper bound on the $P(B_n)$ and $\sum_{k=1}^{\infty} P(A_k)$ is an upper bound on the $P(B_n)$.

Thus $P\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} P(A_k)$.

§1.4

p46 15] If $P(A) > 0$, the conditional probability of B given A is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = P_A(B).$$

If $(\Omega, \mathcal{F}, P)$ is a probability space, $(A, \mathcal{F}_A, P_A)$ is a probability space.

when ~~two~~ not defining 𝔉

p46

16] ~~know~~ For $A_1, A_2, \ldots$

$\cup$ second

$$\overline{\lim}\, A_n = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \{\omega : \omega \in A_n \text{ for infinitely many } A_n\}$$

$\cap$ second

$$\underline{\lim}\, A_n = \liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \{\omega : \omega \in A_n \text{ for all but finitely many } n\}$$

If $A_n \in \mathcal{F}$ then $\overline{\lim}\, A_n, \underline{\lim}\, A_n \in \mathcal{F}$.

17) $\omega \in \limsup_n A_n$ iff for each n, $\exists\, k \geq n \ni \omega \in A_k$ iff ω is in infinitely many of the A_n. If n has the role of time, A_n occurs infinitely often.

p47 18] $\omega \in \liminf_n A_n$ iff $\exists n$

$\ni \omega \in A_k \ \forall k \geq n$ iff ω lies in all but finitely many of the A_n.

If n has the role of time, the A_n occur "almost always" = for all but finitely many n.

19] $\bigcap_{k=n}^{\infty} A_k \uparrow \liminf_n A_n$

$\bigcup_{k=n}^{\infty} A_k \downarrow \limsup_n A_n$.

For every m and n

$$\bigcap_{k=m}^{\infty} A_k \subseteq \bigcup_{k=n}^{\infty} A_k$$

([illegible] of the RHS)

Since for $i \geq \max(m, n)$, $LHS \subseteq A_i \subseteq RHS$.

Taking union over m and intersection over n shows

$$\liminf_n A_n \subseteq \limsup_n A_n.$$

Also if ω lies in all but

finitely many of the A_n ($\omega \in \liminf_n A_n$)

then ω lies in infinitely many A_n ($\omega \in \limsup_n A_n$).

p47 20] Know If $\limsup_n A_n = \liminf_n A_n$,

then $\lim_n A_n = \limsup_n A_n = \liminf_n A_n$.

A_n has limit A, written $A_n \to A$ (converges to)

if $\lim_n A_n = A$.

If $\lim_n A_n = A$ and $A_n \in \mathcal{F}$, then $A \in \mathcal{F}$.

If $\limsup_n A_n \subseteq A \subseteq \liminf_n A_n$,

then $\lim_n A_n = A$.

p48 21] Know Th 4.1 For each sequence $\{A_n\}$ of $\mathcal{F}$ sets

i) $P(\liminf_n A_n) \leq \liminf_n P(A_n) \leq \limsup_n P(A_n)$

$\leq P(\limsup_n A_n)$.

ii) If $A_n \to A$, then $P(A_n) \to P(A)$
(Continuity of prob).

proof ii) follows from i).

($P(A_n) \to P(A)$ means $\forall \varepsilon > 0$, $\exists N > 0 \ni n > N \Rightarrow |P(A_n) - P(A)| < \varepsilon$.)

* $P(A_n) \to P(A)$ iff $\overline{\lim} P(A_n) = \underline{\lim} P(A_n)$

(To prove i), we need to show $\overline{\lim}$ and $\underline{\lim}$ can be exchanged with P. This idea is "the same as continuity": $\lim f(x_n) = f(\lim x_n)$.)

We need to show $P(\underline{\lim} A_n) \le \liminf_n P(A_n)$ and $\overline{\lim} P(A_n) \le P(\overline{\lim} A_n)$.

skip ↓ (Recall $\overline{\lim}\, x_n = \limsup x_n = \inf\limits_{n} \sup\limits_{k \ge n} x_k$

$\underline{\lim}\, x_n = \liminf x_n = \sup\limits_{n} \inf\limits_{k \ge n} x_k$

inf = infimum = greatest lower bound
sup = supremum = least upper bound.

$L = \overline{\lim}\, x_n$ iff i) given $\varepsilon > 0$ $\exists n \ni x_k \le L + \varepsilon \ \forall k \ge n$

Prob 7

ii) given ε and given n, $\exists k \geq n \ni$

$x_k > L - \varepsilon$.

SKIP ↑

Fact: $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$.

Let $B_n = \bigcap_{k=n}^{\infty} A_k \uparrow \liminf_n A_n$

and $C_n = \bigcup_{k=n}^{\infty} A_k \downarrow \limsup_n A_n$.

Then $P(A_n) \geq P(B_n) \to P(\liminf A_n)$

and $P(A_n) \leq P(C_n) \to P(\limsup A_n)$.

Want to take limits, $\lim P(B_n) = P(\liminf_n A_n)$

by monotone continuity, but (continuity from below)

can't take $\lim P(A_n)$ since

$\underline{\lim}\, P(A_n)$ may not equal $\overline{\lim}\, P(A_n)$.

We can take $\underline{\lim}$ of both sides

and $a_n \leq b_n \ \forall n \Rightarrow \underline{\lim}\, a_n \leq \underline{\lim}\, b_n$.

(*) So $\underline{\lim}\, P(A_n) \geq \liminf P(B_n) = P(\underline{\lim}\, A_n)$.

Fact: $(\limsup_n A_n)^c = \liminf_n A_n^c$.

Thus $1 - P(\limsup_n A_n) = P[(\limsup_n A_n)^c]$

$= P(\liminf_n A_n^c) \underset{\uparrow \text{ by } (*)}{\leq} \liminf_n P(A_n^c)$ 75

$= 1 - \limsup_n P(A_n)$.

Thus $P(\limsup_n A_n) \geq \limsup_n P(A_n)$.

$\therefore$ ii) and i) hold.

ex] $A_n = \{(-1)^n\}$. $\limsup_n A_n = \{-1, 1\}$
$\liminf_n A_n = \phi$

22] Def] Given a prob space $(\Omega, \mathcal{F}, P)$, an <u>event</u> A is any $A \in \mathcal{F}$.

Note: Typically $\mathcal{F}$ is not the class of all subsets of Ω. (Some authors define an event to be any subset of Ω.)

p48 23] i) Two events A and B are <u>independent</u>, $A \perp\!\!\!\perp B$, if $P(A \cap B) = P(A)P(B)$.

ii) A finite collection of events $A_1, \dots, A_n$ is <u>independent</u> if for <u>any</u> subcollection $A_{i_1}, \dots, A_{i_k}$, $P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} P(A_{i_j})$.

iii) An infinite (perhaps uncountable) collection of events is <u>independent</u> if <u>each</u> of its finite subcollections is ind.