countably many points since F and G are cdfs.

Hence at all points by right continuity.

$\square$

Know

26] Th: Let $g: \mathbb{R} \to \mathbb{R}$ be a continuous function that does not depend on n.

a) If $X_n \xrightarrow{D} X$, then $g(X_n) \xrightarrow{D} g(X)$.

b) If $X_n \xrightarrow{P} X$, then $g(X_n) \xrightarrow{P} g(X)$.

c) If $X_n \xrightarrow{wp1} X$, then $g(X_n) \xrightarrow{wp1} g(X)$.

Note a) is the continuous mapping theorem. proved soon.

p340 know

27] Th: Suppose the X_n and X are RVs on the same probability space.

If $X_n \xrightarrow{wp1} X$, then $X_n \xrightarrow{P} X$.

If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{D} X$.

Note: in 27] If $X_n \xrightarrow{wp1} X$, then $X_n \xrightarrow{D} X$.

proof of 26 c)

If $X_n(\omega) \to X(\omega)$, then $g(X_n(\omega)) \to g(X(\omega))$.

Since $X_n \xrightarrow{ae} X$, $g(X_n) \xrightarrow{ae} g(X)$, eg on a set A where $P(A) = 1$.

28] <u>Helly-Bray-Portmanteau Th</u>

$X_n \xrightarrow{D} X$ iff $E[g(X_n)] \to E[g(X)]$

for every bounded, real, continuous function g.

<u>Proof of continuous mapping Th</u> If g is real and continuous, then $\cos[t\,g(x)]$ and $\sin[t\,g(x)]$ are bounded real continuous functions.

Hence by the above theorem, for each $t \in \mathbb{R}$

acts as a constant, $t\,g(x)$ is contin

$$\varphi_{g(X_n)}(t) = E\left[e^{it\,g(X_n)}\right] =$$

$$E\left(\cos[t\,g(X_n)]\right) + i\,E\left(\sin[t\,g(X_n)]\right)$$

$$\to E\left(\cos[t\,g(X)]\right) + i\,E\left(\sin[t\,g(X)]\right)$$

$$= E\left[e^{it\,g(X)}\right]$$

$= \varphi_{g(X)}(t)$. $\therefore$ $g(X_n) \xrightarrow{D} g(X)$ by the continuity th. $\square$

29] Notes for proving the CLT:

a) $\varphi_X(t) = 1 + \sum_{j=1}^{K} \frac{E[X^j]}{j!}(it)^j + o(t^k)$ ← little o

is a Taylor series expansion of φ_X provided

$E[X^k]$ exists. where the (error) remainder

$R(t) = o(t^k)$ satisfies $\frac{o(t^k)}{t^k} \to 0$ as $t \to 0$.

b) $\left(1 - \frac{a}{n}\right)^n \to e^{-a}$ $\therefore$ $\left(1 - \frac{a \pm \varepsilon}{n}\right)^n \to e^{-[a \pm \varepsilon]}$

and if $\varepsilon \to 0$, then $\left(1 - \frac{a \pm \varepsilon}{n}\right)^n \to e^{-a}$
even if ε is complex valued.

c) $\varphi_{aX}(t) = E\left(e^{itaX}\right) = \varphi_X(at)$.

If $\varphi_{X_n}(t) \to \varphi_X(t)$ for every t,

then at is a t^* and $\varphi_{X_n}(at) = \varphi_{aX_n}(t)$

$\to \varphi_X(at) = \varphi_{aX}(t)$ for every t.

Hence if $Z_n = \frac{\sqrt{n}\,(\bar{Y} - \mu)}{\sigma} = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}$ satisfies

$\varphi_{Z_n}(t) \to \varphi_Z(t) = e^{-t^2/2}$, the $N(0,1)$ char fn,

then $\sigma Z_n = \sqrt{n}(\overline{Y} - \mu)$ has $\varphi_{\sigma Z_n}(t) \to \varphi_{\sigma Z}(t) =$

$\varphi_Z(\sigma t) = e^{-\sigma^2 t^2/2}$, the $N(0, \sigma^2)$ char fn,

and $\sigma Z_n \xrightarrow{D} N(0, \sigma^2)$. So the CLT holds.

$\boxed{\text{Proof of the CLT}}$ $Z_n = \dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \underbrace{\dfrac{n}{n}}_{1}$

$= \dfrac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma\sqrt{n}} = \dfrac{\sum_{i=1}^{n} (Y_i - \mu)}{\sigma\sqrt{n}}$ where the

$Y_i - \mu$ are iid with char fn $\varphi_{Y-\mu}(t)$.

Hence the char fn of $\dfrac{Y_i - \mu}{\sigma\sqrt{n}}$ is $\varphi_{Y-\mu}\left(\dfrac{t}{\sigma\sqrt{n}}\right)$

and the char fn of Z_n is $\varphi_{Z_n}(t) \overset{\text{iid}}{=} \left[\varphi_{Y-\mu}\left(\dfrac{t}{\sigma\sqrt{n}}\right)\right]^n$.

$E[Y_i - \mu] = 0$ and $E(Y_i - \mu)^2 = \sigma^2$, so by

29a) $\varphi_{Y-\mu}(t) = 1 - \dfrac{\sigma^2}{2} t^2 + o(t^2)$

and $\varphi_{Y-\mu}\left(\dfrac{t}{\sigma\sqrt{n}}\right) = 1 - \dfrac{t^2}{2n} + o\left(\dfrac{t^2}{n}\right)$.

(Cramer p 90 Rao p 90, 107)

where $\dfrac{o(\frac{t^2}{n})}{\frac{t^2}{n}} \to 0$ as $n \to \infty$.

Hence $n\, o(\frac{t^2}{n}) \to 0$ as $n \to \infty$.

Thus $\varphi_{Z_n}(t) = \left[1 - \frac{t^2}{2n} + o(\frac{t^2}{n})\right]^n =$

$$\left[1 - \frac{t^2/2 - n\, o(\frac{t^2}{n})}{n}\right]^n \to e^{-t^2/2} = \varphi_Z(t) \quad \forall t$$

by 29b). $\therefore$ $Z_n \xrightarrow{D} Z \sim N(0,1)$ by the continuity theorem, and $\sigma Z_n = \sqrt{n}(\bar{Y}_n - \mu) \xrightarrow{D} N(0,\sigma^2)$ by 29c).

$\square$

30] Let $\mathbb{C}$ be the complex numbers. If $c_n \to c \in \mathbb{R}$, then
Let $c_n \in \mathbb{R}$ or $\mathbb{C}$.
$\lim_{n\to\infty} \left(1 + \frac{c_n}{n}\right)^n = e^c$.

p352 31] Some properties of characteristic functions]
i) $\varphi(0) = 1$
ii) $|\varphi(t)| \le 1$ ← modulus $\quad \forall t$ where $|a+ib| = \sqrt{a^2+b^2}$.

iii) φ is continuous in t

iv) $\varphi(t)$ is uniformly continuous.

v) $\varphi^{(k)}(0) = i^k \, E(X^k)$ if $E(X^k)$ is finite.

32] Hence if $\varphi_n(t) \to h(t)$ and $h(t)$ is not continuous, then X_n does not converge in distribution to X by the continuity theorem.
See HW8.

33] If X has mgf $m(t)$ (exists for $-\delta < t < \delta$ for some $\delta > 0$), then $E(X^k) < \infty \quad \forall \ k \in \mathbb{N} = \{1, 2, \ldots\}$.

34] Let $j, k \in \mathbb{N}$. If $E[X^k]$ is finite, then $E(X^j)$ is finite for $1 \leq j \leq k$.

Proof] If $|y| \leq 1$, then $|y^j| = |y|^j \leq 1$ and if $|y| > 1$, then $|y|^j \leq |y|^k$.

Hence $|y|^j \leq |y|^k + 1$ and $|X|^j \leq |X|^k + 1$, $X = X(\omega)$

$\therefore \ E[|X|^j] \leq E[|X|^k] + 1 < \infty$.

35] The CLT for iid X_i is known as the Lindeberg-Lévy CLT.

36] $c(t) = \log[M(t)]$ = cumulant generating function
$c'(0) = E(X)$, $c''(0) = V(X)$
L'Hôspital's rule: suppose $f(x) \to 0$ and $g(x) \to 0$ as $\underbrace{x \downarrow d,\ x \uparrow d,\ x \to d,\ x \to \infty \text{ or } x \to -\infty}$.
If $\frac{f'(x)}{g'(x)} \to L$, then $\frac{f(x)}{g(x)} \to L$ as ↑.

If the mgf exists, then $E(X^k)$ exists $\forall k \in \mathbb{N}$.

37] Proof of CLT if mgf exists.
Let $Y_1, Y_2, \ldots$ be iid with mean μ, variance σ^2 and mgf $m_Y(t)$ for $|t| < t_o$. Then
$Z_i = \frac{Y_i - \mu}{\sigma}$ has mean 0, variance 1 and
mgf $M_Z(t) = \exp\left(\frac{-t\mu}{\sigma}\right) m_Y\left(\frac{t}{\sigma}\right)$ for $|t| < \sigma t_o$.

want to show $W_n = \sqrt{n}\left(\frac{\overline{Y}_n - \mu}{\sigma}\right) \xrightarrow{D} N(0,1)$.

$$W_n = n^{-\frac{1}{2}} \sum_{i=1}^n Z_i = n^{-\frac{1}{2}} \sum_{i=1}^n \left(\frac{Y_i - \mu}{\sigma}\right) =$$

$$n^{-\frac{1}{2}} \frac{\sum_{i=1}^n Y_i - n\mu}{\sigma} = \frac{n^{-\frac{1}{2}}}{\frac{1}{n}} \frac{\overline{Y}_n - \mu}{\sigma}$$

$\therefore M_{W_n}(t) = E(e^{tW_n}) = E\left[\exp\left(t\, n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_i\right)\right]$

$= E\left[\exp\left(\sum_{i=1}^{n} \frac{tZ_i}{\sqrt{n}}\right)\right] \overset{\text{ind}}{=} \prod_{i=1}^{n} E\left[e^{tZ_i/\sqrt{n}}\right]$

$\overset{\text{iid}}{=} \prod_{i=1}^{n} M_Z\left(\frac{t}{\sqrt{n}}\right) = \left[M_Z\left(\frac{t}{\sqrt{n}}\right)\right]^n$. Let

$C_Z(x) = \log[M_Z(x)]$. Then $C_{W_n}(t) =$

$\log[M_{W_n}(t)] = n \log\left[M_Z\left(\frac{t}{\sqrt{n}}\right)\right] = n\, C_Z\left(\frac{t}{\sqrt{n}}\right)$

$= \dfrac{C_Z\left(\frac{t}{\sqrt{n}}\right)}{\frac{1}{n}}$. Now $C_Z(0) = \log[M_Z(0)] = \log(1) = 0$.

$\therefore$ by L'Hôspital's rule (deriv wrt n)

$$\lim_{n\to\infty} \log[M_{W_n}(t)] = \lim_{n\to\infty} \frac{C_Z\left(\frac{t}{\sqrt{n}}\right)}{\frac{1}{n}} =$$

$$\lim_{n\to\infty} \frac{C_Z'\left(\frac{t}{\sqrt{n}}\right)\left(\frac{-t}{2n^{3/2}}\right)}{\frac{-1}{n^2}} = \frac{t}{2} \lim_{n\to\infty} \frac{C_Z'\left(\frac{t}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}}.$$

Now $C_Z'(0) = \dfrac{M_Z'(0)}{M_Z(0)} = \dfrac{E(Z)}{1} = 0.$

So L'Hôspital's rule can be applied again:

$$\lim_{n\to\infty} \log[M_{W_n}(t)] = \frac{t}{2} \frac{C_Z''\left(\frac{t}{\sqrt{n}}\right)\left(\frac{-t}{2n^{3/2}}\right)}{\left(\frac{-1}{2n^{3/2}}\right)} =$$

$$\frac{t^2}{2} \lim_{n\to\infty} C_Z''\left(\frac{t}{\sqrt{n}}\right) = \frac{t^2}{2} \underbrace{C_Z''(0)}_{V(Z)}$$

$$\underset{\uparrow}{=} \frac{t^2}{2} V(Z_i) = \frac{t^2}{2}.$$

E1 rev 25)

Thus $\lim_{n\to\infty} \log[M_{W_n}(t)] = \frac{t^2}{2}$ and

$\lim_{n\to\infty} M_{W_n}(t) = e^{t^2/2}$, the $N(0,1)$ mgf.

$\therefore W_n = \sqrt{n}\left(\frac{\bar{Y}_n - \mu}{\sigma}\right) \xrightarrow{D} N(0,1)$

by the continuity th. $\square$

Remark: there is a proof for the CLT using similar techniques with $\varphi_Z(t)$.

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38] Prove $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$

proof] $F_n(x) = P(X_n \le x) = \overbrace{P(X_n \le x, X > x+\varepsilon) + P(X_n \le x, X \le x+\varepsilon)}^{\text{partition}}$

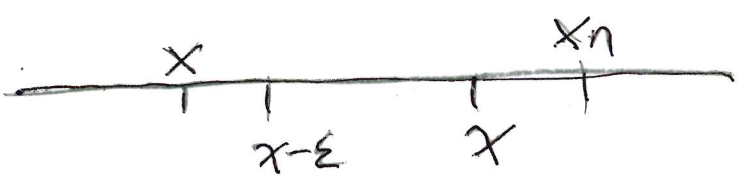
$$\le P(|X_n - X| \ge \varepsilon) + P(X \le x+\varepsilon)$$

$$= P(|X_n - X| \ge \varepsilon) + F_X(x+\varepsilon).$$

$F_X(x-\varepsilon) = P(X \le x-\varepsilon) =$

$$\overbrace{P(X \le x-\varepsilon, X_n > x) + P(X \le x-\varepsilon, X_n \le \varepsilon)}^{\text{partition}}$$

$$\le P(|X_n - X| \ge \varepsilon) + P(X_n \le x)$$



$$= P(|X_n - X| \ge \varepsilon) + F_n(x).$$

$$\therefore F_X(x-\varepsilon) - P(|X_n - X| \ge \varepsilon) \le F_n(x) \le P(|X_n - X| \ge \varepsilon) + F_X(x+\varepsilon).$$