Since $X_n \xrightarrow{P} X$, $P(|X_n - X| \geq \varepsilon) \to 0$ as $n \to \infty$.

If $F_X(x)$ is continuous at x, then

$F_X(x-\varepsilon) \to F_X(x)$ and $F_X(x+\varepsilon) \to F_X(x)$ as $\varepsilon \to 0$.

Taking limsup and liminf gives

$F_X(x-\varepsilon) \leq \liminf F_n(x) \leq \limsup F_n(x) \leq F_X(x+\varepsilon)$.

$\therefore F_n(x) \to F_X(x)$ if $F_X(x)$ is contin at x.

$\square$ Ash stat

39] a) If $\underbrace{E[(X_n - X)^2] \to 0}_{X_n \xrightarrow{2} X}$ as $n \to \infty$, then $X_n \xrightarrow{P} X$.

b) If $E[X_n] \to E(X)$ and $V(X_n - X) \to 0$, then $X_n \xrightarrow{P} X$.

Know c) If $X \equiv c$, b) is $E[X_n] \to c$, $V(X_n) \to 0 \Rightarrow X_n \xrightarrow{P} c$.

proof: a) By Generalized Chebyshev's Inequality,

$$P(|X_n - X| \geq \varepsilon) = P[(X_n - X)^2 \geq \varepsilon^2] \leq \frac{E[(X_n - X)^2]}{\varepsilon^2} \to 0 \text{ as } n \to \infty.$$

$E[W^2] = V(W) + [E(W)]^2$

b) $E[(X_n - X)^2] = V(X_n - X) + [E(X_n - X)]^2$

$= V(X_n - X) + [E(X_n) - E(X)]^2 \to 0$

as $n \to \infty$.

$\square$

$$P[W^2 \geq \varepsilon^2] \leq \frac{E(W^2)}{\varepsilon^2}$$

65.5

40] $X_n \xrightarrow{P} c \iff X_n \xrightarrow{D} c$

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proof] showed $X_n \xrightarrow{P} c \Rightarrow X_n \xrightarrow{D} c$ in 38].

$$P[|X_n - c| \geq \varepsilon] = P(X_n \geq c+\varepsilon) + P(X_n \leq c-\varepsilon)$$

$$= 1 - P(X_n < c+\varepsilon) + P(X_n \leq c-\varepsilon) = RHS$$

Now $P(X_n < c+\varepsilon) \geq P(X_n \leq c+\frac{\varepsilon}{2})$. $\therefore$

$$P[|X_n - c| \geq \varepsilon] = RHS \leq 1 - P(X_n \leq c+\tfrac{\varepsilon}{2}) + P(X_n \leq c-\varepsilon)$$

$$= 1 - \underbrace{F_n(c+\tfrac{\varepsilon}{2})}_{\to 1} + \underbrace{F_n(c-\varepsilon)}_{\to 0},$$ justification

$F_n(t) \to F_X(t)$ for $t \neq c$ where $F_X(t) = \begin{cases} 1 & t \geq c \\ 0 & t < c. \end{cases}$

$\therefore P[|X_n - X| \geq \varepsilon] \to 1 - 1 + 0 = 0$ as $n \to \infty$.

$\square$

41] p346 RVs X_n are <u>tight</u> or <u>bounded in probability</u> if for each $\varepsilon > 0$,

$\exists\, x_\varepsilon$ and y_ε $\ni$ $F_n(x_\varepsilon) < \varepsilon$ and $F_n(y_\varepsilon) > 1-\varepsilon$ $\forall n \geq 1$,

iff $\forall \varepsilon > 0\ \exists$ constants D_ε and N_ε $\ni$

$P(|X_n| \leq D_\varepsilon) \geq 1-\varepsilon$ $\forall n \geq N_\varepsilon$. (Increase

D_ε to D'_ε so $P(|X_n| \le D'_\varepsilon) \ge 1-\varepsilon$ $\forall\, n \ge 1$.)

If X_n is not tight, then "mass escapes to ∞ or $-\infty$."

42] p346 Tightness is a necessary and sufficient condition that for every sequence $\{F_{n_k}\}$ there exists a further subsequence $\{F_{n_{k(j)}}\}$ such that $X_{n_{k(j)}} \xrightarrow{D} X_j$ for some RV X_j.

Remark] Hence if the X_n are not tight, X_n can't converge in distribution to some RV X. (Take $\{F_n\}$ to be the seq.)

If $X_n \xrightarrow{D} X$, then the X_n are tight.

43] If $n^\delta (X_n - c) \xrightarrow{D} X$ where $0 < \delta \le 1$, then $X_n \xrightarrow{P} c$. In particular, if $\sqrt{n}(X_n - c) \xrightarrow{D} N(0,\sigma^2)$, $X_n \xrightarrow{P} c$.

44] Slutsky's Theorem: If $Y_n \xrightarrow{D} Y$ and $W_n \xrightarrow{P} c$,

then a) $Y_n + W_n \xrightarrow{D} Y + c$

b) $Y_n W_n \xrightarrow{D} cY$

c) $Y_n / W_n \xrightarrow{D} \frac{Y}{c}$ if $c \neq 0$.

Note: $c_n \to c$ as $n \to \infty \Rightarrow c_n \xrightarrow{ae} c \Rightarrow c_n \xrightarrow{P} c$.

Note: $W_n \xrightarrow{P} c \Leftrightarrow W_n \xrightarrow{D} c$.

45] If $X_n \xrightarrow{r} X$, then $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{D} X$.

Showed $X_n \xrightarrow{2} X \Rightarrow X_n \xrightarrow{P} X$.

46] $X_n \xrightarrow{r} X \Rightarrow X_n \xrightarrow{k} X$ $\quad 0 < k < r$.

proof of 45] $|X_n - X|^r \geq |X_n - X|^r \underbrace{I(|X_n - X| \geq \varepsilon)}_{1 \text{ or } 0}$

$\geq \varepsilon^r I[|X_n - X| \geq \varepsilon]$

So for any $\varepsilon > 0$,

$$E[|X_n - X|^r] \geq E[|X_n - X|^r I(|X_n - X| \geq \varepsilon)]$$

$$\geq E[\varepsilon^r I(|X_n - X| \geq \varepsilon)]$$

$$= \varepsilon^r P(|X_n - X| \geq \varepsilon).$$

So $P[|X_n - X| \geq \varepsilon] \leq \dfrac{E[|X_n - X|^r]}{\varepsilon^r} \to 0$ as $n \to \infty$.

$\square$

47) $X_n \xrightarrow{P} X \Rightarrow X_n$ is close to X.

$X_n \xrightarrow{D} X$ does not imply that X_n is close to X.

Jensen's Inequality:

48) (Hw 2) $g(E(X)) \leq E[g(X)]$ if the expected values exist and g is convex on an interval containing the range of X.

49) Proof that $X_n \xrightarrow{r} X \Rightarrow X_n \xrightarrow{k} X$, $0 < k < r$. see p44

Let $U_n = |X_n - X|^r$ and $W_n = |X_n - X|^k$.

Then $U_n = W_n^t$ where $t = \frac{r}{k} > 1$ and $g(x) = x^t$ is convex on $[0, \infty)$. By Jensen's inequality,

$$E[|X_n - X|^r] = E[U_n] = E[W_n^t]$$

$$\geq (E[W_n])^t = \left(E[|X_n - X|^k]\right)^{\frac{r}{k}}, \quad r > k.$$

$\therefore \lim_{r \to \infty} E[|X_n - X|^r] = 0 \Rightarrow \lim_{n \to \infty} E[|X_n - X|^k] = 0$, $0 < k < r$.

$\square$

50) Let X_n have pdf $f_{X_n}(x)$ and let X have pdf $f_X(x)$.

P350 problem 25.9

a) If $f_{X_n}(x) \to f_X(x)$ $\forall x$, then $X_n \xrightarrow{D} X$.

b) If $f_{X_n}(x) \to f_X(x)$ $\underbrace{\text{ae } \underbrace{\lambda}_{\text{Lebesgue measure}}}_{\text{for } x \text{ outside a set of Lebesgue measure } 0}$,

then $X_n \xrightarrow{D} X$.

51) suppose $g: \mathbb{R} \to \mathbb{R}$ is continuous at c.

a) If $X_n \xrightarrow{D} c$, $g(X_n) \xrightarrow{D} c$

b) If $X_n \xrightarrow{P} c$, $g(X_n) \xrightarrow{P} c$

c) If $X_n \xrightarrow{wP1} c$, $g(X_n) \xrightarrow{wP1} c$.

52) Suppose X_n and X are integer valued RVs.

Then $X_n \xrightarrow{D} X$ iff $P(X_n = k) \to P(X = k)$ for every integer k.

iff $\underbrace{P_{X_n}(x) \to P_X(x)}_{\text{pmfs}}$ $\forall$ real x

DasGupta

53] Billingsley's proof of CLT

$$\varphi_X(t) = E[e^{itX}] = E[\cos(tX)] + i\,E[\sin(tX)]$$

$|\varphi_X(t)| \leq 1 \quad \forall t$ since

even for complex RV

$$E[e^{itX}] \leq E[|e^{itX}|] = E[1]$$

$$e^{itx} = \cos(tx) + i\sin(tx)$$

so $|e^{itx}| = \sqrt{\cos^2(tx) + \sin^2(tx)} = \sqrt{1}$

Lemma: Let $z_1, \ldots, z_m$ and $w_1, \ldots, w_m$ be complex numbers of modulus at most 1. Then

$$|(z_1 \cdots z_m) - (w_1 \cdots w_m)| \leq \sum_{k=1}^{m} |z_k - w_k|.$$

proof: see HW 8.

Let $Z_n = \sqrt{n}\left(\frac{\bar{Y}_n - \mu}{\sigma}\right) = \sum_{i=1}^{n} \frac{Y_i - \mu}{\sigma\sqrt{n}}$

$$\varphi_{Z_n}(t) = \left[\varphi_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n.$$

$$\varphi_{Y-\mu}(t) = 1 - \frac{\sigma^2}{2}t^2 + R(t), \qquad R(t) = O(t^2), \; \frac{R(t)}{t^2} \to 0 \text{ as } t^2 \to 0$$

$$\varphi_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 - \frac{t^2}{2n} + R\left(\frac{t}{\sigma\sqrt{n}}\right)$$

So $\left|\varphi_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)-\left(1-\frac{t^2}{2n}\right)\right|=\left|R\left(\frac{t}{\sigma\sqrt{n}}\right)\right|=o\left(\frac{t^2}{\sigma^2 n}\right)$

$$\frac{R\left(\frac{t}{\sigma\sqrt{n}}\right)}{\frac{t^2}{\sigma^2 n}}\to 0 \text{ so } \left|\frac{R\left(\frac{t}{\sigma\sqrt{n}}\right)}{\frac{t^2}{\sigma^2 n}}\right|\to 0$$

$$n\, o\left(\frac{t^2}{\sigma^2 n}\right)\to 0 \quad \therefore$$

$$n\left|\varphi_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)-\left(1-\frac{t^2}{2n}\right)\right|\to 0.$$

By lemma with n so large that $0<\frac{t^2}{2n}<1<2$,

$z_i=\varphi_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)$, $w_i=1-\frac{t^2}{2n}$,

$$\left|\left[\varphi_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n-\left(1-\frac{t^2}{2n}\right)^n\right|\le n\left|\varphi_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)-\left(1-\frac{t^2}{2n}\right)\right|\to 0.$$

$$\therefore \varphi_{Z_n}(t)=\left[\varphi_{Y-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n\to \lim_n\left(1-\frac{t^2/2}{n}\right)^n=\underbrace{e^{-t^2/2}}_{\varphi_Z(t),\ Z\sim N(0,1)}\ \forall t$$

$$\therefore Z_n\xrightarrow{D}N(0,1).$$

karr p109

54] A complex RV Z has the form $Z = X + iY$ where X and Y are ordinary RVs. $E[Z] = E[X] + i\,E(Y)$, and Z is integrable if $E[|Z|] = E[\sqrt{X^2+Y^2}] < \infty$.

Linearity, LDCT and key inequalities remain valid, including $|E[Z]| \leq E[|Z|]$.

$Z = e^{itX}$ is the main complex RV in this class.

55) p360 Th If $\lim_{n \to \infty} \varphi_{X_n}(t) = g(t)$ $\forall t$ where g is continuous at $t=0$, then $g(t) = \varphi_X(t)$ is a characteristic function and $X_n \xrightarrow{D} X$.

Note: Hence continuity at $t=0 \Rightarrow$ continuity everywhere since $g(t) = \varphi_X(t)$ is continuous.
If $g(t)$ is not continuous at $t=0$, then X_n does not converge in distribution.

p27 56) For each n, let $W_{n1}, \ldots, W_{nr_n}$ be independent.
The probability space may change with n.

Let $E[W_{nk}] = 0$, $V(W_{nk}) = E[W_{nk}^2] = \sigma_{nk}^2$

and $\Delta_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2 = V\left[\sum_{k=1}^{r_n} W_{nk}\right]$. $Z_n = \dfrac{\sum_{k=1}^{r_n} W_{nk}}{\Delta_n}$ is the z-score of $\sum_{k=1}^{r_n} W_{nk}$.

57] Lyapounov's CLT: Under 56],

assume the $|W_{nk}|^{2+\delta}$ are integrable for some $\delta > 0$. Assume Lyapounov's condition: $\lim_n \sum_{k=1}^{r_n} \dfrac{E[|W_{nk}|^{2+\delta}]}{\Delta_n^{2+\delta}} = 0$.

$$\text{Then } Z_n = \frac{\sum_{k=1}^{r_n} W_{nk}}{\Delta_n} \xrightarrow{D} N(0,1).$$

58] Special cases: i) $r_n = n$, $W_{nk} = W_k$ has $W_1, \ldots, W_n, \ldots$ independent.

ii) $W_{nk} = X_{nk} - E(X_{nk}) = X_{nk} - \mu_{nk}$

$$\text{has } \frac{\sum_{k=1}^{r_n} (X_{nk} - \mu_{nk})}{\Delta_n} \xrightarrow{D} N(0,1).$$

iii) Suppose $X_1, X_2, \ldots$ are ind

with $E(X_i) = \mu_i$, $V(X_i) = \sigma_i^2$.

Let $Z_n = \dfrac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i}{\left(\sum_{i=1}^{n} \sigma_i^2\right)^{\frac{1}{2}}}$ be the

Z score of $\sum_{i=1}^{n} X_i$. Hence $E(Z_n) = 0$, $V(Z_n) = 1$.

Assume $E\left[|X_i - \mu_i|^3\right] < \infty$ for $i \in \mathbb{N}$

and $$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} E\left[|X_i - \mu_i|^3\right]}{\left(\sum_{i=1}^{n} \sigma_i^2\right)^{3/2}} = 0. \quad (*)$$

Then $Z_n \xrightarrow{D} N(0,1)$.

Proof of iii): Take $W_{nk} = X_k - \mu_k$, $\delta = 1$

$s_n = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$ and apply Lyapounov's CLT.

Note that $\left(\sum_{i=1}^{n} \sigma_i^2\right)^{3/2} = (s_n^2)^{3/2} = s_n^3 = s_n^{2+1}$.

59] The (Lindeberg - Lévy) CLT has the X_i iid with $V(X_i) = \sigma^2 < \infty$.
The Lyapounov CLT in 58 iii) has the X_i independent (not necessarily identically distributed) with $E[|X_i|^3] < \infty$ and satisfies (*).

ex] Dual problem
suppose the $X_i \overset{ind}{\sim} Ber(p_i)$ and

$$\sum_{i=1}^{\infty} p_i q_i = \infty \quad \text{where } q_i = 1 - p_i. \quad V(X_i) = p_i q_i.$$

claim $Z_n = \dfrac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} p_i}{\left(\sum_{i=1}^{n} p_i q_i\right)^{\frac{1}{2}}} \xrightarrow{D} N(0,1)$.

x	1	0
$p(x)$	p_i	q_i

$X_i - p_i = W$	$1-p_i$	$-p_i$
$P(w)$	p_i	q_i

proof: Let $Y_i = |W_i| = |X_i - p_i|$

x	1	0
y	$1-p_i$	p_i
$P(y)$	p_i	q_i

$E[|X_i - p_i|^3] =$

$$E[|W_i|^3] = E[Y_i^3] = \sum_y y^3 P(y) = (1-p_i)^3 p_i + p_i^3 q_i$$

$$= q_i^3 p_i + p_i^3 q_i = p_i q_i \underbrace{(p_i^2 + q_i^2)}_{\leq (p_i + q_i)^2 = 1} \leq p_i q_i$$