$$\therefore \sum_{i=1}^{n} E\left[|X_i - p_i|^3\right] \le \sum_{i=1}^{n} p_i q_i$$

divide both sides by $\left(\sum p_i q_i\right)^{3/2}$

or
$$\frac{\sum_{i=1}^{n} E\left[|X_i - p_i|^3\right]}{\left(\sum_{i=1}^{n} p_i q_i\right)^{3/2}} \le \frac{1}{\left(\sum_{i=1}^{n} p_i q_i\right)^{\frac{1}{2}}} \to 0$$

$\therefore$ (*) in 58 iii) holds and $Z_n \xrightarrow{D} N(0,1)$.

Degroot

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60] <u>Lindeberg CLT</u>: Let the W_{nk} satisfy 56) and <u>Lindeberg's condition</u>

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{s_n^2} E\left(W_{nk}^2 \, I\left[|W_{nk}| \ge \varepsilon s_n\right]\right)$$

$= 0$ for any $\varepsilon > 0$. Then

$Z_n \xrightarrow{D} N(0,1)$.

Note: Sometimes called the Lindeberg–Feller CLT.

61] Lindeberg's condition is

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{|W_{nk}| \ge \varepsilon s_n} W_{nk}^2 \, dP = 0 \quad \text{for any } \varepsilon > 0.$$

62] Special case: Assume $r_n = n$ and the $W_{nk} = W_k$ are independent.

(*) If $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{s_n^2} E\left(W_k^2 \, I[|W_k| \ge \varepsilon s_n]\right) = 0 \quad \forall \varepsilon > 0,$

then $Z_n = \dfrac{\sum_{k=1}^{n} W_k}{s_n} \xrightarrow{D} N(0,1).$

63] Lindeberg's condition is nearly necessary for $Z_n = \dfrac{\sum_{k=1}^{r_n} W_{nk}}{s_n} \xrightarrow{D} N(0,1).$

64] a) uniformly bounded sequence

Let $r_n = n$, $W_{nk} = W_k$. If there is a constant c such that $P(|W_k| < c) = 1 \quad \forall k,$

and if $s_n \to \infty$, then 62) holds.

proof: Once n is large enough so that

$\underbrace{\varepsilon s_n > c}_{\text{occurs since } s_n \to \infty}$, $I[|W_k| \ge \varepsilon s_n] = 0$ so (*) $= 0$.

p369 b) If $W_1, W_2, \ldots$ are iid with $V(W_i) = \sigma^2 \in (0,\infty)$ then 62) holds.

proof: $s_n^2 = n\sigma^2$ so

$$\frac{1}{s_n^2} \sum_{k=1}^{n} E\Big[\underbrace{W_k^2\, I\big[|W_k| \ge \varepsilon s_n\big]}_{\text{iid for given } n}\Big]$$

$$= \frac{1}{\sigma^2} E\Big[\underbrace{W_1^2\, I\big[|W_1| \ge \varepsilon \sigma \sqrt{n}\big]}_{X_n \downarrow 0 = X \text{ as } n \uparrow \infty}\Big] \underset{\text{take } Y = W_1^2}{\to 0 \text{ by LDCT.}}$$

$$\left(= \frac{1}{\sigma^2} \int_{\{|W_1| \ge \varepsilon\sigma\sqrt{n}\}} W_1^2 \, dP \to 0 \quad \text{as } n \uparrow \infty \right.$$

$$\left. \text{since } P(|W_1| \ge \varepsilon\sigma\sqrt{n}) \downarrow 0 \text{ as } n \uparrow \infty. \right)$$

65] Taking $W_i = X_i - \mu$ means 64b)
$\Rightarrow$ (Lindeberg Lévy) CLT holds.

66] If the Lyapounov condition holds, then Lindeberg's condition holds. So the Lindeberg CLT proves the Lyapounov CLT.

proof) $$\sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{\{|W_{nk}| \geq \varepsilon s_n\}} W_{nk}^2 \, dP \leq$$

$|W_{nk}|^\delta \geq \varepsilon^\delta s_n^\delta$ on the set

so $\frac{|W_{nk}|^\delta}{\varepsilon^\delta s_n^\delta} \geq 1$ on the set

$$\sum_{k=1}^{r_n} \frac{1}{s_n^2} \int_{\{|W_{nk}| \geq \varepsilon s_n\}} \frac{|W_{nk}|^{2+\delta}}{\varepsilon^\delta s_n^\delta} \, dP$$

$$\leq \frac{1}{\varepsilon^\delta} \sum_{k=1}^{r_n} \frac{1}{s_n^{2+\delta}} E\left[|W_{nk}|^{2+\delta}\right] \to 0 \quad \forall \varepsilon > 0 \text{ if}$$

Lyapounov's condition holds.

$$\int_A \underbrace{|W_{nk}|^{2+\delta}}_{\geq 0} dP \leq \int_\Omega |W_{nk}|^{2+\delta} dP = E\left[|W_{nk}|^{2+\delta}\right]$$

$\Omega = A \cup A^c$, $\int_\Omega |f| dP = \int_A |f| dP + \int_{A^c} |f| \, dP$

§ 5.29

1) Change in notation

Let $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} \in \mathbb{R}^k$ be a column vector.

$$E[\underline{X}] = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_k) \end{bmatrix}$$

$$\text{cov}(\underline{X}) = \Sigma = E\left[(\underline{X} - E(\underline{X}))(\underline{X} - E(\underline{X}))^T\right] = (\sigma_{ij})$$

where $\sigma_{ij} = cov(X_i, X_j)$ and $cov(X_i, X_i) = \sigma_i^2 = V(X_i)$.

2] p395 The characteristic function of $\underline{X}$ is

$$\varphi_{\underline{X}}(\underline{t}) = E\left[e^{i\underline{t}^T \underline{X}}\right]$$

The moment generating function of $\underline{X}$ is

$$M_{\underline{X}}(\underline{t}) = E\left[e^{\underline{t}^T \underline{X}}\right]$$ provided the expectation exists for all $\underline{t}$ in a neighborhood of $\underline{0}$.

The cdf $F_{\underline{X}}(\underline{x}) = P(X_1 \le x_1, \ldots, X_n \le x_n)$.

3] Let $\|x\| = \sqrt{x^T x} = \sqrt{x_1^2 + \cdots + x_k^2}$.

4] Def. Let $\underline{X}_n \in \mathbb{R}^k$ be a sequence of random vectors RVs with joint cdfs $F_{\underline{X}_n}(\underline{x})$ and let $\underline{X} \in \mathbb{R}^k$ be a RV with cdf $F_{\underline{X}}(\underline{x})$.

p390 a) $\underline{X}_n$ converges in distribution to X, $\underline{X}_n \xrightarrow{D} \underline{X}$, if $F_{\underline{X}_n}(\underline{x}) \to F_{\underline{X}}(\underline{x})$ as $n \to \infty$ for all continuity points $\underline{x}$ of $F_{\underline{X}}(\underline{x})$. $\underline{X}$ is the limiting or asymptotic dist of $\underline{X}_n$, and does not depend on n.

b) $\underset{\sim}{X_n}$ converges in probability to $\underline{X}$,

$\underset{\sim}{X_n} \xrightarrow{P} \underline{X}$, if $P\left[\|\underline{X_n} - \underline{X}\| \geq \varepsilon\right] \to 0$ as $n \to \infty$

$\forall \varepsilon > 0$.

c) Let $r > 0$. $\underset{\sim}{X_n}$ converges in rth mean to $\underset{\sim}{X}$

$\underset{\sim}{X_n} \xrightarrow{r} \underset{\sim}{X}$, if $E\left[\|\underline{X_n} - \underline{X}\|^r\right] \to 0$ as $n \to \infty$.

d) $\underset{\sim}{X_n}$ converges almost everywhere to $\underline{X}$

almost surely

with probability 1

$\underline{X_n} \xrightarrow{ae} \underline{X}$, $\underline{X_n} \xrightarrow{as} \underline{X}$, $\underline{X_n} \xrightarrow{wp1} \underline{X}$, if

$P\left(\lim_{n \to \infty} \underline{X_n} = \underline{X}\right) = 1$.

For $\xrightarrow{B} \underset{\sim}{c}$, replace $\underset{\sim}{X}$ by $\underset{\sim}{c}$

$B = ae, P, r$ or D

5) Generalized Chebyshev's Theorem:

Let $\underset{K \times 1}{\underset{\sim}{X}} \in \mathbb{R}^K$ and $U : \mathbb{R}^K \to [0, \infty)$ be a nonnegative function. Then for any $c > 0$

$$P\left[U(\underline{X}) \geq c\right] \leq \frac{E\left[U(\underline{X})\right]}{c}.$$

proof when $\underline{X}$ has pdf $f_{\underline{X}}(\underline{x})$:

$$E[U(\underline{X})] = \int_{\mathbb{R}^k} U(\underline{x}) f(\underline{x})\, d\underline{x}$$

column vector

row vectors

$$\geq \int_{\{\underline{x}:\, U(\underline{x}) \geq c\}} U(\underline{x}) f(\underline{x})\, d\underline{x}$$

$\left(\int_{\{\underline{x}:\, U(\underline{x}) < c\}} U(\underline{x}) f(\underline{x})\, d\underline{x} \geq 0 \right.$ since $\left. U(\underline{x}) f(\underline{x}) \geq 0 \right)$

$$\geq c \int_{\{\underline{x}:\, U(\underline{x}) \geq c\}} f(\underline{x})\, d\underline{x} = c\, P(U(\underline{X}) \geq c)$$

$\square$

6] Let $E(\underline{X}) = \underline{\mu}$ and $U(\underset{\sim}{x}) = \|\underset{\sim}{x} - \underset{\sim}{\mu}\|$.

Then $P[\|\underline{X} - \underline{\mu}\| \geq c] = P[\|\underset{\sim}{X} - \underset{\sim}{\mu}\|^r \geq c^r]$ for $c > 0$.

$$\leq \frac{E[\|\underline{X} - \underline{\mu}\|^r]}{c^r}.$$

"Markov's Inequality"

can replace $\underline{\mu}$ by $\underset{\sim}{a}$

7] Let $\underset{\sim}{X}$ and $\underline{Y}$ be $k \times 1$ RVs, $\underline{a}$ a conformable constant vector and A and B conformable constant matrices.

i) $E[\underline{a} + \underline{X}] = \underline{a} + E(\underline{X})$

ii) $E[\underline{X} + \underline{Y}] = E[\underline{X}] + E(\underline{Y})$

has the right dimension

iii) $E(A\underset{\sim}{x}) = A E(\underset{\sim}{x})$

iv) $E[A \underset{\sim}{x} B] = \underset{q\times k}{A}\ \underset{k\times 1}{E(\underset{\sim}{x})}\ \underset{1\times p}{B}$

v) $\text{cov}(\underset{\sim}{a} + A\underset{\sim}{x}) = \text{cov}(A\underset{\sim}{x}) = A\,\text{cov}(\underset{\sim}{x})A^T$.

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8] $\underset{k\times 1}{\underset{\sim}{X}}$ has a k-dimensional multivariate normal distribution $N_k(\underset{\sim}{\mu}, \Sigma)$ if $\underset{\sim}{t}^T\underset{\sim}{x}$ has a univariate normal distribution for any $k\times 1$ constant vector $\underset{\sim}{t}$.

univariate normal
$k=1$ $w \sim N(\mu, \sigma^2)$

$E[\underset{\sim}{x}] = \underset{\sim}{\mu}$, $\text{cov}(\underset{\sim}{x}) = \Sigma$.

$\underset{\sim}{t}^T\underset{\sim}{x} \sim N(\underset{\sim}{t}^T\underset{\sim}{\mu}, \underset{\sim}{t}^T\Sigma\underset{\sim}{t})$.

(write N_1 as N)

9] If $\underset{\sim}{x} \sim N_k(\underset{\sim}{\mu}, \Sigma)$ and A is a $q\times k$ constant matrix, b a constant, $\underset{\sim}{a}$ a $k\times 1$ constant vector, $\underset{\sim}{d}$ a $q\times 1$ constant vector, then

i) $A\underset{\sim}{x} \sim N_q(A\underset{\sim}{\mu}, A\Sigma A^T)$.

ii) $\underset{\sim}{a} + b\underset{\sim}{x} \sim N_k(\underset{\sim}{a} + b\underset{\sim}{\mu}, b^2\Sigma)$.

Note $b\underset{\sim}{x} = b I_k \underset{\sim}{x}$ with $A = b I_k$.

iii) $A\underset{\sim}{x} + \underset{\sim}{d} \sim N_q(A\underset{\sim}{\mu} + \underset{\sim}{d}, A\Sigma A^T)$.

10] Suppose $\underset{\sim}{X}_n \xrightarrow{D} \underset{\sim}{X} \sim N_k(\mu, \Sigma)$.

Then i) $A\underset{\sim}{X}_n \xrightarrow{D} A\underset{\sim}{X} \sim N_q(A\underset{\sim}{\mu}, A\Sigma A^T)$

ii) $\underset{\sim}{a} + b\underset{\sim}{X}_n \xrightarrow{D} N_k(\underset{\sim}{a} + b\underset{\sim}{\mu}, b^2\Sigma)$

iii) $A\underset{\sim}{X}_n + \underset{\sim}{d} \xrightarrow{D} N_q(A\underset{\sim}{\mu} + \underset{\sim}{d}, A\Sigma A^T)$

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11) Multivariate Central Limit Theorem (MCLT):

If $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ are iid $k \times 1$ random vectors with $E(\underset{\sim}{x}) = \underset{\sim}{\mu}$ and $\text{cov}(\underset{\sim}{x}) = \Sigma$, then

$$\sqrt{n}\,(\overline{\underset{\sim}{X}}_n - \underset{\sim}{\mu}) \xrightarrow{D} N_k(\underset{\sim}{0}, \Sigma).$$

Note: $\overline{\underset{\sim}{X}}_n = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{x}_i$.

usual CLT is a special case with $k = 1$.

ex] Suppose the $\underset{\sim}{X}_i$ are iid $p \times 1$ random vectors with $E(\underset{\sim}{x}_i) = e^{0.5}\underset{\sim}{1}$ and $\text{cov}(\underset{\sim}{x}_i) = (e^2 - e) I_p$. Find the limiting distribution of $\sqrt{n}(\overline{X}_n - \underset{\sim}{c})$ for appropriate $\underset{\sim}{c}$.

soln] $\sqrt{n}\,(\overline{X}_n - e^{0.5}\underset{\sim}{1}) \xrightarrow{D} N_p[\underset{\sim}{0}, (e^2 - e) I_p]$

common error: get p wrong

See Quiz 10
HW 10

12] If $0 < \delta \le 1$ and

$n^{\delta}(\underset{\sim}{X}_n - \underset{\sim}{c}) \xrightarrow{D} \underline{X}$, then

$\underline{X}_n \xrightarrow{P} \underset{\sim}{c}$.

13] Th: If $\underline{X}_1, \ldots, \underline{X}_n$ are iid with $E(\underline{X}) = \underline{\mu} \in \mathbb{R}^k$

a) WLLN: $\overline{\underset{\sim}{X}}_n \xrightarrow{P} \underline{\mu}$

b) SLLN: $\overline{\underset{\sim}{X}}_n \xrightarrow{wp1} \underline{\mu}$.

p396
14] Continuity Theorem: Let $\underline{X}_n, \underset{\sim}{X}$ be $k \times 1$ with char functions $\varphi_{\underline{X}_n}(\underline{t})$ and $\varphi_{\underline{X}}(\underline{t})$.

$\underline{X}_n \xrightarrow{D} \underline{X}$ iff $\varphi_{\underline{X}_n}(\underline{t}) \to \varphi_{\underline{X}}(\underline{t})$ $\forall \underline{t} \in \mathbb{R}^k$.

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15] Th: Cramér Wold Device Let $\underline{X}_n, \underline{X}$ be $k \times 1$.

$\underline{X}_n \xrightarrow{D} \underline{X}$ iff $\underline{t}^T \underline{X}_n \xrightarrow{D} \underline{t}^T \underline{X}$ $\forall \underline{t} \in \mathbb{R}^k$.

16] know Proof of MCLT: For fixed $\underline{t}$, the $\underline{t}^T \underline{X}_i$ are iid with mean $\underline{t}^T \underline{\mu}$ and variance $\underline{t}^T \Sigma \underline{t}$. Hence by the CLT,

$$\underline{t}^T \sqrt{n}(\overline{\underline{X}}_n - \underline{\mu}) \xrightarrow{D} \underline{t}^T \underline{X} \sim N(\underbrace{0}_{\text{scalar}}, \underbrace{\underline{t}^T \Sigma \underline{t}}_{\text{scalar}})$$

where $\underline{X} \sim N_k(\underline{0}, \Sigma)$. Hence by the Cramér Wold device, $\sqrt{n}(\overline{\underline{X}}_n - \underline{\mu}) \xrightarrow{D} N_k(\underline{0}, \Sigma)$.

$\square$

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17) <u>Continuous mapping Th</u>: Let $\underline{X}, \underline{X}_n \in \mathbb{R}^k$. If $\underline{X}_n \xrightarrow{D} \underline{X}$ and if $\underline{g}: \mathbb{R}^k \to \mathbb{R}^j$ is continuous, then $\underline{g}(\underline{X}_n) \xrightarrow{D} \underline{g}(\underline{X})$.

18] Th: Let $\underline{X}_n = (X_{1n}, ..., X_{kn})^T$, let $\underline{Y}_n \in \mathbb{R}^k$, let $\underline{X} = (X_1, ..., X_k)^T$. Let W_n be a sequence of $k \times k$ nonsingular random matrices and let C be a $k \times k$ constant nonsingular matrix.

a) $\underline{X}_n \xrightarrow{P} \underline{X}$ iff $X_{in} \xrightarrow{P} X_i$ for $i = 1, ..., k$.

b) <u>Slutsky's Th</u>: If $\underline{X}_n \xrightarrow{D} \underline{X}$, $\underline{Y}_n \xrightarrow{P} \underline{c}$, and if $W_n \xrightarrow{D} C$, then for some constant vector $\underline{c}$,

i) $\underline{X}_n + \underline{Y}_n \xrightarrow{D} \underline{X} + \underline{c}$

ii) $\underline{Y}_n^T \underline{X}_n \xrightarrow{D} \underline{c}^T \underline{X}$

iii) $W_n \underline{X}_n \xrightarrow{D} C \underline{X}$

iv) $\underline{X}_n^T W_n \xrightarrow{D} \underline{X}^T C$

v) $W_n^{-1} \underline{X}_n \xrightarrow{D} C^{-1} \underline{X}$

vi) $\underline{X}_n^T W_n^{-1} \xrightarrow{D} \underline{X}^T C^{-1}$.

19] If $\underline{X}_n \xrightarrow{D} \underline{X}$, then $X_{in} \xrightarrow{D} X_i$ for $i = 1, \dots, K$.

proof: use the cramér wold device with $\underline{t}_i = [0, \dots, 0, 1, 0, \dots, 0]^T$ (↑ ith)

$\therefore \underline{t}_i^T \underline{X}_n = X_{in} \xrightarrow{D} X_i = \underline{t}_i^T \underline{X}$. $\Box$

20] In general, $\underline{X}_{in} \xrightarrow{D} \underline{X}_i$ for $i = 1, \dots, m$

<u>does not imply</u> $\begin{pmatrix} \underline{X}_{1n} \\ \vdots \\ \underline{X}_{mn} \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \underline{X}_1 \\ \vdots \\ \underline{X}_m \end{pmatrix}$. That is,

marginal convergence in distribution does not imply joint convergence in distribution.

21] Independence is an exception since independent RVs have a joint dist. that does not affect the marginal dist.
Suppose $\underline{X}_n \perp\!\!\!\perp \underline{Y}_n$ for $n = 1, 2, \dots$.
Suppose $\underline{X}_n \xrightarrow{D} \underline{X}$ and $\underline{Y}_n \xrightarrow{D} \underline{Y}$ where $\underline{X} \perp\!\!\!\perp \underline{Y}$. Then $\begin{pmatrix} \underline{X}_n \\ \underline{Y}_n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \underline{X} \\ \underline{Y} \end{pmatrix}$.

proof] Let $\underline{t} = \begin{pmatrix} \underline{t}_1 \\ \underline{t}_2 \end{pmatrix}$, $\underline{Z}_n = \begin{pmatrix} \underset{\sim}{X}_n \\ \underset{\sim}{Y}_n \end{pmatrix}$ and $\underline{Z} = \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix}$,

Since $\underline{X}_n \perp\!\!\!\perp \underline{Y}_n$ and $\underline{X} \perp\!\!\!\perp \underline{Y}$,

$$\varphi_{\underline{Z}_n}(\underline{t}) = \varphi_{\underline{X}_n}(\underline{t}_1)\, \varphi_{\underline{Y}_n}(\underline{t}_2) \to \varphi_{\underline{X}}(\underline{t}_1)\, \varphi_{\underline{Y}}(\underline{t}_2) = \varphi_{\underline{Z}}\begin{pmatrix} \underline{t}_1 \\ \underline{t}_2 \end{pmatrix}.$$

Hence $\underline{Z}_n \xrightarrow{D} \underline{Z}$ by the continuity theorem

and $\underline{g}(\underline{Z}_n) \xrightarrow{D} \underset{\sim}{g}(\underline{Z})$ ~~for~~ continuous $\underline{g}$

by the continuous mapping theorem.

77.5

22) Th: Let $\underline{X} = \begin{pmatrix} \underline{X}_1 \\ \underline{X}_2 \end{pmatrix} \in \mathbb{R}^k$

with $\underline{X}_1 \in \mathbb{R}^{k_1}$, $\underline{X}_2 \in \mathbb{R}^{k_2}$ and $k_1 + k_2 = k$

Let $\varphi_{\underline{X}}$, $\varphi_{\underline{X}_1}$ and $\varphi_{\underline{X}_2}$ be the char fns of $\underline{X}$, $\underline{X}_1$ and $\underline{X}_2$. Then $\underline{X}_1 \perp\!\!\!\perp \underline{X}_2$ iff

$$\varphi_{\underline{X}}(\underline{t}) = \varphi_{\underline{X}_1}(\underline{t}_1)\, \varphi_{\underline{X}_2}(\underline{t}_2) \quad \forall\, \underline{t} = \begin{pmatrix} \underline{t}_1 \\ \underline{t}_2 \end{pmatrix} \in \mathbb{R}^k.$$

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Prove $\underline{X}_1 \perp\!\!\!\perp \underline{X}_2 \Rightarrow \varphi_{\underline{X}}(\underline{t}) = \varphi_{\underline{X}_1}(\underline{t}_1)\, \varphi_{\underline{X}_2}(\underline{t}_2)$:

$$\varphi_{\underline{X}}(\underline{t}) = E\left(e^{i\underline{t}^T\underline{X}}\right) = E\left[e^{i(\underline{t}_1^T\ \underline{t}_2^T)\begin{pmatrix} \underline{X}_1 \\ \underline{X}_2 \end{pmatrix}}\right]$$

$$= E\left[e^{i(\underline{t}_1^T\underline{X}_1 + \underline{t}_2^T\underline{X}_2)}\right] = E\left[e^{i\underline{t}_1^T\underline{X}_1}\, e^{i\underline{t}_2^T\underline{X}_2}\right]$$

$$\underline{\underline{\text{ind}}}\ E\left[e^{i\underline{t}_1^T\underline{X}_1}\right] E\left[e^{i\underline{t}_2^T\underline{X}_2}\right] = \varphi_{\underline{X}_1}(\underline{t}_1)\, \varphi_{\underline{X}_2}(\underline{t}_2).$$

$\square$

23] If $\underline{X}_n \xrightarrow{D} \underline{X}$ and $\underline{Y}_n \xrightarrow{P} \underline{c}$, a constant vector, then $\begin{pmatrix} \underline{X}_n \\ \underline{Y}_n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \underline{X} \\ \underline{c} \end{pmatrix}$.

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