ex] Let $X \sim N(0,1)$, $W \sim N(0,1)$,

$X_n = X \quad \forall n$, and

$Y_n = -X \quad \forall n$.

Then $X_n \xrightarrow{D} W$ and

$Y_n \xrightarrow{D} W$.

Thus $\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \not\xrightarrow{D} \begin{pmatrix} W \\ W \end{pmatrix}$

Could take $Y_n = \begin{cases} X & n \text{ odd} \\ -X & n \text{ even} \end{cases}$

So $X_n + Y_n = \begin{cases} 2X & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$\therefore X_n + Y_n \not\xrightarrow{D}$

$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} X \\ -X \end{pmatrix} \quad \forall n$

$\therefore \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{D} \begin{pmatrix} X \\ -X \end{pmatrix}$

$X_n + Y_n \xrightarrow{D} 0 \sim N(0,0)$

Since $(1 \ \ 1)\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = X_n + Y_n \underset{\text{for all } n}{=} 0 \not\xrightarrow{D} (1 \ \ 1)\begin{pmatrix} W \\ W \end{pmatrix}$

$= 2W \sim N(0,4)$.

24] Th] i) $\underline{X}_n \xrightarrow{wp1} \underline{X} \Rightarrow \underline{X}_n \xrightarrow{P} \underline{X}$

ii) $\underline{X}_n \xrightarrow{r} \underline{X} \Rightarrow \underline{X}_n \xrightarrow{P} \underline{X}$

iii) $\underline{X}_n \xrightarrow{P} \underline{X} \Rightarrow \underline{X}_n \xrightarrow{D} \underline{X}$.

iv) $\underline{X}_n \xrightarrow{P} \underline{c}$ iff $\underline{X}_n \xrightarrow{D} \underline{c}$

proof ii) a) Let $\varepsilon > 0$.

$E\left[\|\underline{X}_n - \underline{X}\|^r\right] \geq E\left[\|\underline{X}_n - \underline{X}\|^r \, I\left(\|\underline{X}_n - \underline{X}\| \geq \varepsilon\right)\right]$

$$\geq \varepsilon^r \; P[\|\underline{X}_n - \underline{X}\| \geq \varepsilon]$$

$$\text{So } P(\|\underline{X}_n - \underline{X}\| \geq \varepsilon) \leq \frac{E[\|\underline{X}_n - \underline{X}\|^r]}{\varepsilon^r} \to 0 \text{ as } n \to \infty.$$

Proof ii) b) $P[\|\underline{X}_n - \underline{X}\| \geq \varepsilon] = P[\|\underline{X}_n - \underline{X}\|^r \geq \varepsilon^r]$

$$\leq \frac{E[\|\underline{X}_n - \underline{X}\|^r]}{\varepsilon^r} \to 0 \text{ by Gen Cheb.}$$

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25) Proof of the Cramer Wold Device.

$$\varphi_{\underbrace{\underline{t}^T \underline{X}_n}_{W}}(y) = E\left[e^{iy\underline{t}^T \underline{X}_n}\right] = \varphi_{\underline{X}_n}(y\underline{t}), \quad y \in \mathbb{R}.$$

and $\varphi_{\underline{t}^T \underline{X}}(y) = \varphi_{\underline{X}}(y\underline{t})$, $y \in \mathbb{R}$.

If $\underline{X}_n \xrightarrow{D} \underline{X}$, then $\varphi_{\underline{X}_n}(\underline{t}) \to \varphi_{\underline{X}}(\underline{t}) \;\; \forall \underline{t} \in \mathbb{R}^k$.

Fix $\underline{t}$. Then $\varphi_{\underline{X}_n}(y\underline{t}) \to \varphi_{\underline{X}}(y\underline{t}) \;\; \forall y \in \mathbb{R}$.

$\therefore \underline{t}^T \underline{X}_n \xrightarrow{D} \underline{t}^T \underline{X}$.

If $\underline{t}^T \underline{X}_n \xrightarrow{D} \underline{t}^T \underline{X} \;\; \forall \underline{t} \in \mathbb{R}^k$, then

$\varphi_{\underline{X}_n}(y\underline{t}) \to \varphi_{\underline{X}}(y\underline{t}) \quad \forall y \in \mathbb{R}, \forall \underline{t} \in \mathbb{R}^k$.

Take $y=1$ to get $\varphi_{\underline{X}_n}(\underline{t}) \to \varphi_{\underline{X}}(\underline{t}) \; \forall \underline{t} \in \mathbb{R}^k$.

$\therefore \underline{X}_n \xrightarrow{D} \underline{X}$. $\square$

Conditional Expectation

§6.32 1] Let μ and ν be measures on
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$(\Omega, \mathcal{F})$. Then <u>ν is absolutely continuous wrt μ</u> if for each $A \in \mathcal{F}$, $\mu(A)=0$ $\Rightarrow \nu(A)=0$, written $\nu << \mu$.

ex] If $\nu(A) = \int_A f \, d\mu$ where f is a ~~pdf~~ nonnegative function, then $\nu << \mu$.

2] p443 <u>Radon-Nikodym Theorem</u>: If μ and ν are σ-finite measures such that $\nu << \mu$, then there exists a nonnegative measurable f, a density, such that

$\nu(A) = \int_A f\,d\mu$ for all $A \in \mathcal{F}$.

For two such densities f and g,

$\mu[f \neq g] = 0$. $\quad$ ($f = g$ μ ae)

Notes: A pdf is a density (often wrt Leb meas).
When measure $\nu(A) = \int_A f\,d\mu$ with f a nonnegative function on $(\Omega, \mathcal{F}, \mu)$, then f is a density. Recall that functions as integrands are assumed to be measurable.
The density $f = \frac{d\nu}{d\mu}$ is called the Radon-Nikodym derivative of ν wrt μ.

§33 <u>conditional probability</u>

3) For $(\Omega, \mathcal{F}, P)$, $\quad P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, $\quad P(B) > 0$.

Goal: Find conditional prob's wrt a σ-field $\mathcal{G} \subseteq \mathcal{F}$.