PM 80

4] Fix A in $\mathcal{F}$. A

<u>conditional probability of A given $\mathcal{G}$</u> is $f = P[A|\mathcal{G}]$

that is i) measurable $\mathcal{G}$, integrable

and ii) $\int_G P[A|\mathcal{G}]\,dP = E\left[P(A|\mathcal{G})\, I_G\right] = P(A \cap G)$

for any $G \in \mathcal{G}$.

5] i) $f = P(A|\mathcal{G})$ is a random variable wrt $\mathcal{G}$.

ii) Let $\nu(G) = \int_G P(A|\mathcal{G})\,dP = P(A \cap G)$, $G \in \mathcal{G}$.

$P(G) = 0 \Rightarrow \nu(G) = 0$. So the Radon-Nikodym theorem can be applied to measures ν and P wrt $(\Omega, \mathcal{G})$. ($\mathcal{G} \subseteq \mathcal{F}$ so P is defined on $\mathcal{G}$.) Hence $P[A|\mathcal{G}] \geq 0$.
$0 \leq P[A|\mathcal{G}] \leq 1$ wp1.

iii) There are many such RVs $P(A|\mathcal{G})$ satisfying 4], but any two of them are equal with prob 1. A specific such RV is called a <u>version</u> of the conditional probability. (unique up to sets of prob 0.)

80.5

iv) It is possible that $P(A)=0$.

p452
ex] Suppose $A \in \mathcal{G}$. Then I_A is $\mathcal{G}$ measurable and $\int_G I_A \, dP = \int I_A I_G \, dP = \int I_{A\cap G} \, dP = P(A \cap G)$. $\therefore$ I_A is a version of $P(A|\mathcal{G})$ if $A \in \mathcal{G}$.

p452
ex] $\mathcal{G} = \{\phi, \Omega\}$. The only $\mathcal{G}$ measurable functions are constant functions. ← these are $\mathcal{G}$ meas. Let $f = P(A)$. Then $\int_\phi P(A) \, dP = 0 = P(A \cap \phi)$

and $\int_\Omega P(A) \, dP = P(A \cap \Omega) = P(A)$.

$\therefore$ $P(A)$ is a version of $P(A|\mathcal{G})$.

p453
ex] Let $A \perp\!\!\!\perp \mathcal{G}$. and $f = P(A)$.

$\int_G P(A) \, dP = P(A) \int I_G \, dP = P(A) P(G)$

$\underset{A \perp\!\!\!\perp G}{=} P(A \cap G)$. $\therefore$ $P(A)$ is a version of $P(A|\mathcal{G})$.

§34 Conditional Expectation

6] p466 (Let $E[X]$ exist on $(\Omega, \mathcal{F}, P)$.) The <u>conditional expectation of X given $\mathcal{G}$</u> is $f = E[X|\mathcal{G}]$ that is i) measurable $\mathcal{G}$, integrable and ii)

$$\int_G f\,dP = \int_G E[X|\mathcal{G}]\,dP = E[E(X|\mathcal{G})\,I_G] = E[X I_G]$$
$$= \int_G X\,dP \quad \text{for } G \in \mathcal{G}.$$

Here X is an integrable RV on $(\Omega, \mathcal{F}, P)$.

7] Define ν on $\mathcal{G}$ for nonnegative X by $\nu(G) = \int_G X\,dP$. Then $\nu \ll P$ and by the Radon Nikodym theorem, there is a function f, measurable $\mathcal{G}$, such that $\nu(G) = \int_G f\,dP = \int_G X\,dP$ for $G \in \mathcal{G}$.

If X is integrable, $f = E[X^+|\mathcal{G}] - E[X^-|\mathcal{G}]$ works. not necessarily nonnegative

8] There are many such RVs $E[X|\mathcal{G}]$. Any one of them is a version of $E[X|\mathcal{G}]$ and any two versions are equal wp1.

Fix A in $\mathcal{F}$.

9) If $X = I_A$, then $E[I_A|\mathcal{G}]$ is a version of $P[A|\mathcal{G}]$.

proof) $\int_G E[I_A|\mathcal{G}]\,dP = \int_G I_A\,dP = \int I_A I_G\,dP$

$= \int I_{A\cap G}\,dP = P(A\cap G) \quad \forall G \in \mathcal{G}$.

ex) If $\mathcal{G} = \mathcal{F}$, X is a version of $E(X|\mathcal{F})$

Since $\int_G X\,dP = \int_G X\,dP \quad \forall G \in \mathcal{F} = \mathcal{G}$.

ex) If $\mathcal{G} = \{\phi, \Omega\}$, only constants are measurable $\mathcal{G}$. Let $f = E(X)$.

$\int_\phi E(X)\,dP = E(X)\,P(\phi) = 0 = \int_\phi X\,dP$

$\int_\Omega E(X)\,dP = E(X)\,P(\Omega) = E(X) = \int_\Omega X\,dP$

$\therefore$ $E(X)$ is a version of $E[X|\mathcal{G}]$.

eg X a constant

10) If $\mathcal{G} \subseteq \mathcal{F}$, then often X is not measurable $\mathcal{G}$, and hence X is not a version of $E[X|\mathcal{G}]$. If X is measurable $\mathcal{G}$, then X is a version of $E[X|\mathcal{G}]$.

11] If X is measurable $\mathcal{G}$, then PM 82
X acts like a constant given $\mathcal{G}$.

p469
12] Th: If X is measurable $\mathcal{G}$ and Y and XY are integrable, then
$E[XY|\mathcal{G}] = X\,E[Y|\mathcal{G}]$ wp1.
That is, $X\,E[Y|\mathcal{G}]$ is a version of $E[XY|\mathcal{G}]$.

p468
13] Th: Suppose X, Y, X_n are integrable.
i) If $X = a$ wp1, then $E[X|\mathcal{G}] = a$ wp1.
ii) For constants a and b,
$E[(aX+bY)|\mathcal{G}] = a\,E[X|\mathcal{G}] + b\,E[Y|\mathcal{G}]$ wp1.
iii) If $X \le Y$ wp1, then $E[X|\mathcal{G}] \le E[Y|\mathcal{G}]$ wp1.
iv) $|E[X|\mathcal{G}]| \le E[|X|\,|\mathcal{G}]$ wp1. redundant 12
v) If $\lim X_n = X$ wp1, $|X_n| \le Y$, and Y is integrable, then $\lim_{n\to\infty} E[X_n|\mathcal{G}] = E[X|\mathcal{G}]$ wp1.

some Proofs i) a is measurable $\mathcal{G}$

and $\int_G a\,dP = \int_G X\,dP = a\,P(G)$ $\forall G \in \mathcal{G}$.

ii) $a\,E[X|\mathcal{G}] + b\,E[Y|\mathcal{G}]$ is measurable $\mathcal{G}$ and integrable. $\therefore$

$\int_G (a\,E[X|\mathcal{G}] + b\,E[Y|\mathcal{G}])\,dP =$

$a\int_G E[X|\mathcal{G}]\,dP + b\int_G E[Y|\mathcal{G}]\,dP =$

$a\int_G X\,dP + b\int_G Y\,dP = \int_G (aX+bY)\,dP$ $\forall G \in \mathcal{G}$.

So $a\,E[X|\mathcal{G}] + b\,E[Y|\mathcal{G}]$ is a version of $E[(aX+bY)\,|\,\mathcal{G}]$.

iii) If $X \leq Y$ wp1, then

$\int_G (\underbrace{E[Y|\mathcal{G}] - E[X|\mathcal{G}]}_{\text{measurable } \mathcal{G}})\,dP = \int_G (Y-X)\,dP \geq 0$ $\forall G \in \mathcal{G}$.

Let $g = E[Y|\mathcal{G}] - E[X|\mathcal{G}]$. Then $\{\omega : g(\omega) < 0\}$

$= g^{-1}(-\infty, 0) = G_1 \in \mathcal{G}$. $\int_{G_1} g\,dP \geq 0$ and $g < 0$ on G_1

$\Rightarrow P(G_1) = 0$. $\therefore$ $E[Y|\mathcal{G}] - E[X|\mathcal{G}] \geq 0$ wp1.

14) Th If X is integrable and σ fields $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$, then

$$\underbrace{E\left[E[X|\mathcal{G}_2]|\mathcal{G}_1\right]}_{LHS} = E[X|\mathcal{G}_1] \quad wp1.$$

Proof: $W = E[X|\mathcal{G}_2]$ is a RV wrt $\mathcal{G}_2$, so

$E[W|\mathcal{G}_1] = LHS$ is measurable $\mathcal{G}_1$.

Let $G \in \mathcal{G}_1$. Then $G \in \mathcal{G}_2$. Thus

$$\int_G E\big[\underbrace{E[X|\mathcal{G}_2]}_{W}|\mathcal{G}_1\big]\,dP = \int_G W\,dP$$

$$= \int_G E[X|\mathcal{G}_2]\,dP \overset{G \in \mathcal{G}_2}{=} \int_G X\,dP \quad \forall G \in \mathcal{G}_1.$$

$\therefore$ $E\left[E[X|\mathcal{G}_2]|\mathcal{G}_1\right]$ is a version of $E[X|\mathcal{G}_1]$.

$\square$

ex) Suppose $X \perp\!\!\!\perp \mathcal{G}$.
Then $E(X)$ is a version of $E[X|\mathcal{G}]$.

proof: $E(X)$ is a constant, so measurable $\mathcal{G}$. 83.5

$$\int_G E[X]\,dP = E[X]\int_G dP = E(X)\,P(G) \quad \forall G \in \mathcal{G}$$

while $\int_G X\,dP = \int X\,I_G\,dP = E[X\,I_G]$

$$\underset{ind}{=} E[X]\,E(I_G) = E(X)\,P(G) \quad \forall G \in \mathcal{G}.$$

15) Intuitively, $\mathcal{G}$ is information, so if $\mathcal{G}$ is known, we know whether $G \in \mathcal{G}$ has occurred or not. So $E[X|\mathcal{G}]$ should be the "best guess" of X given information $\mathcal{G}$.

Earlier showed X is a version of $E[X|\mathcal{G}]$ if X is measurable $\mathcal{G}$.

16) Let $E[X\|\mathcal{G}]$ (2 bars) be the class of all $\mathcal{G}$ measurable RVS $E[X|\mathcal{G}]$.
Then $f \in E[X\|\mathcal{G}]$ means f is a version of $E[X|\mathcal{G}]$.

Good exam problem

17)* $\mathcal{G} \subseteq \mathcal{F}$ and X $\mathcal{G}$ measurable $\Rightarrow X^{-1}(B) \in \mathcal{G}$ $\forall B \in \mathcal{B}(\mathbb{R})$
$\Rightarrow X^{-1}(B) \in \mathcal{F}$ $\forall B \in \mathcal{B}(\mathbb{R}) \Rightarrow X$ is $\mathcal{F}$ measurable.

converse is false