§35 Martingales

p480, 484, 485

1) Def. Let $X_1, X_2, \ldots$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \cdots$ be an increasing sequence of σ fields in $\mathcal{F}$.

Assume

M1) X_n is measurable $\mathcal{F}_n$

M2) $E(|X_n|) < \infty$

Then $X_1, X_2, \ldots$ is

M3) a <u>martingale</u> (relative to $\mathcal{F}_1, \mathcal{F}_2, \ldots$)

if $E(X_{n+1} | \mathcal{F}_n) = X_n$ wp1

S3) a <u>submartingale</u> (relative to $\mathcal{F}_1, \mathcal{F}_2, \ldots$)

if $E(X_{n+1} | \mathcal{F}_n) \geq X_n$ wp1

W3) a <u>supermartingale</u> (relative to $\mathcal{F}_1, \mathcal{F}_2, \ldots$)

if $E(X_{n+1} | \mathcal{F}_n) \leq X_n$ wp1.

2) a) $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ is an increasing sequence of σ-fields in $\mathcal{F}$ iff

$$\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \quad \text{for } n = 1, 2, \ldots$$

*b) $E[|X_n|] < \infty$ for each n iff X_n is integrable for each n iff $E(X_n) \in \mathbb{R}$ for each n. (useful for M2)

c) A martingale is both a submartingale and a supermartingale.

d) We may say ~~that~~ $\{(X_n, \mathcal{F}_n)\}$ is a martingale or that $\{X_n\}$ is a martingale

e) wp1 is always understood and may be omitted

f) The σ-fields $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)$ $(= \mathcal{F}_n)$ always work.

Then M3) becomes $E(X_n | \mathcal{G}_n) = \ldots$

$$E(X_n | X_1, \ldots, X_n) = X_n$$

S3) becomes $E(X_n|\mathcal{G}_n) = E(X_n|X_1,\ldots,X_n) \geq X_n$

and U3) becomes $E(X_n|\mathcal{G}_n) = E(X_n|X_1,\ldots,X_n) \leq X_n$.

g) If Z_n is a measurable and integrable function of $X_1,\ldots,X_n$, then $\mathcal{G}_n = \sigma(X_1,\ldots,X_n)$ $= \mathcal{F}_n$ tends to work.

Tip: try to write Z_{n+1} as a function of Z_n and X_{n+1} to show that M3) (or S3) or U3)) holds.

h) If $Z_n = g(X_1,\ldots,X_n)$ where g is a measurable function so that Z_n is a RV, then

$$E(Z_n W|X_1,\ldots,X_n) = Z_n E(W|X_1,\ldots,X_n).$$

i) If $E(W)$ exists and $W \perp\!\!\!\perp (X_1,\ldots,X_n)$, then $E(W|X_1,\ldots,X_n) = E(W)$

j) A submartingale has upward trend.
A supermartingale has downward trend.

3) Recall from [34.1 14],
If X is integrable and $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$,
then $E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[X|\mathcal{G}_1]$ wp1.

Thus $E[E[X|\mathcal{F}_{n+1}]|\mathcal{F}_n] = E[X|\mathcal{F}_n]$ wp1.

4) For a martingale,

$E(X_{n+2}|\mathcal{F}_{n+1}) = X_{n+1}$ since $\underline{M3}$)

is true for each $n \in \mathbb{N}$. typo in online notes

5) Th a) If $\{X_n\}$ is a martingale,

then $E[X_{n+k}|\mathcal{F}_n] = X_n$ for $n, k = 1, 2, \ldots$,

b) If $\{X_n\}$ is a submartingale, then

$E[X_{n+k}|\mathcal{F}_n] \geq X_n$ for $n, k = 1, 2, \ldots$,

c) If $\{X_n\}$ is a supermartingale, then

$E(X_{n+k}|\mathcal{F}_n) \leq X_n$ for $n, k = 1, 2, \ldots$.

Proof a) by 3) $E[X_{n+2}|\mathcal{F}_n] =$

$E[E(X_{n+2}|\mathcal{F}_{n+1})|\mathcal{F}_n] = E(X_{n+1}|\mathcal{F}_n) = X_n$.

Use induction to get the general result.

6] Submartingale Convergence Th:

Let $\{X_n\}$ be a submartingale. If $K = \sup_n E[|X_n|] < \infty$, then $X_n \xrightarrow{wp1} X$ where X is a RV with $E[|X|] \le K$.

7] Fact: $E[X \mid \sigma(X_1, \dots, X_n)] = E(X \mid X_1, \dots, X_n)$

$$E[g(X_1, \dots, X_n) \mid \sigma(X_1, \dots, X_n)]$$

$$= E[g(X_1, \dots, X_n) \mid X_1, \dots, X_n] = g(X_1, \dots, X_n)$$

see 2]

ex] Let $X_1, X_2, \dots$ be independent RVs with $E(X_k) = 0$ for $k = 1, 2, \dots$. Let $S_n = \sum_{i=1}^n X_i = X_1 + \dots + X_n$. Show that $\{S_n\}$ is a martingale

Soln] Tip 1 usually take $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ but could take $\mathcal{F}_n = \sigma(S_1, \dots, S_n)$

Tip 2] Write S_{n+1} as a function of S_n so that S_n is constant given $\mathcal{F}_n$.

0) Take $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

M1) holds since S_n is a function of $X_1, \ldots, X_n$ hence measurable $\mathcal{F}_n$ (finite sums are measurable)

M2) holds $E[|S_n|] \le E|X_1| + \cdots + E|X_n| < \infty$

(or since $E(S_n) = 0$ for each n, see 2b).)

M3 holds since $S_{n+1} = S_n + X_{n+1}$ so

$$E(S_{n+1} \mid \mathcal{F}_n) = E[S_n + X_{n+1} \mid X_1, \ldots, X_n]$$

$$= E[S_n \mid X_1, \ldots, X_n] + E[X_{n+1} \mid \underbrace{X_1, \ldots, X_n}_{ind}]$$

$$= S_n + 0 = S_n$$

$\uparrow$

S_n is a function of $X_1, \ldots, X_n$

ex) Let $X_1, X_2, \ldots$ be independent RVs with $E(X_k) = 0$ and finite variances $\sigma_k^2 = E(X_k^2)$ for $k = 1, 2, \ldots$.

Let $A_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$, and $S_n = X_1 + \cdots + X_n$.

Let $Y_n = S_n^2 - A_n^2$. Show that $\{Y_n\}$ is a martingale

0) Take $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$

M1) holds since Y_n is a RV and a function of $X_1, \ldots, X_n$, hence measurable $\mathcal{F}_n$.

M2) holds since $E|Y_n| \le E S_n^2 + s_n^2$

$= 2 s_n^2$ $\quad$ ($E(S_n) = 0$ so $E(S_n^2) =$

$V(S_n) = s_n^2$). (see 2b).)

To show M3) holds, note that

$S_{n+1}^2 = (S_n + X_{n+1})^2$. $\quad$ So $E(S_{n+1}^2 | X_1, \ldots, X_n)$

$= E\{S_n^2 + 2S_n X_{n+1} + X_{n+1}^2 | X_1, \ldots, X_n\}$

$= E(S_n^2 | X_1, \ldots, X_n) + 2S_n E(X_{n+1} | X_1, \ldots, X_n) + E(X_{n+1}^2 | X_1, \ldots, X_n)$

$= S_n^2 + 2S_n \underbrace{E(X_{n+1})}_{0} + E(X_{n+1}^2) = S_n^2 + \sigma_{n+1}^2.$

Thus $E(Y_{n+1} | X_1, \ldots, X_n) =$

$E\{S_{n+1}^2 - s_{n+1}^2 | X_1, \ldots, X_n\} = E(S_{n+1}^2 | X_1, \ldots, X_n) - s_{n+1}^2$

$= S_n^2 + \sigma_{n+1}^2 - (s_n^2 + \sigma_{n+1}^2) = S_n^2 - s_n^2 = Y_n$

for each n.

ex] Let $X_1, X_2, \ldots$ be iid nonnegative RVs with $E(X_k) = 1$ for $k = 1, 2, \ldots$.

Let $Y_n = \prod_{i=1}^{n} X_i$.

i) Show that $\{Y_n\}$ is a martingale.

ii) Is there a RV Y such that $Y_n \xrightarrow{wp1} Y$?

<u>Good Qual Problem</u>

Soln] i) 0) Take $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

M1) holds since the product Y_n is a function of $X_1, \ldots, X_n$, hence measurable $\mathcal{F}_n$.

M2) holds by 2b) since

$$E|Y_n| = \prod_{i=1}^{n} E(X_i) = 1 < \infty \text{ for } n = 1, 2, \ldots.$$

To show M3) holds note that

$Y_{n+1} = Y_n X_{n+1}$. Thus

$$E(Y_{n+1} | X_1, \ldots, X_n) = E(Y_n X_{n+1} | X_1, \ldots, X_n)$$

$$= Y_n E[X_{n+1} | X_1, \ldots, X_n] = Y_n \underbrace{E(X_{n+1})}_{1} = Y_n \quad \forall n.$$

ii) Since the X_i are nonnegative,

$E\{|Y_n|\} = E(Y_n) = \prod_{i=1}^{n} E(X_i) = 1 = k.$

$(\sup_n E|Y_n| = 1 = k < \infty.)$ Thus

by the submartingale convergence theorem, there does exist RV Y such that $Y_n \xrightarrow{wp1} Y$

(recall that a martingale is a submartingale).

ex) Let $X_1, X_2, \ldots$ be ind RVs with $E(X_k) = \mu_k$ for $k = 1, 2, \ldots$. Let $T_n = \sum_{k=1}^{n} (X_k - \mu_k)$. Show that $\{T_n\}$ is martingale.

0) Take $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$

M1) holds since T_n is a function of $X_1, \ldots, X_n$, hence measurable $\mathcal{F}_n$

M2) holds since $E(T_n) = 0 \in \mathbb{R}\ \forall n$.

To see that M3) holds, note that

$T_{n+1} = T_n + X_{n+1} - \mu_{n+1}$. Thus

$$E(T_{n+1} \mid X_1, \ldots, X_n) = E(T_n + X_{n+1} - \mu_{n+1} \mid X_1, \ldots, X_n)$$

$$= T_n + E(X_{n+1} \mid X_1, \ldots, X_n) - \mu_{n+1} =$$

$$T_n + \underbrace{E(X_{n+1})}_{\mu_{n+1}} - \mu_{n+1} = T_n.$$

8) Suppose that g is a one to one and onto function so that the inverse function g^{-1} exists. eg g is increasing, decreasing, convex or concave (some texts add "strictly"). Then $E(W \mid X_1, \ldots, X_n) = E(W \mid g(X_1), \ldots, g(X_n))$.

9) Th: Suppose $\{X_n\}$ is a martingale.

a) If g is convex, then $\{Z_n = g(X_n)\}$ is a submartingale.

b) If g is concave, then $\{Z_n = g(X_n)\}$ is a supermartingale.

proof: a) Using a version of Jensen's

Inequality adapted to conditional expectations,

$E(Z_n \mid Z_1, \ldots, Z_n) = E[g(X_{n+1}) \mid g(X_1), \ldots, g(X_n)]$

$\underset{\text{by 8)}}{=} E[g(X_{n+1}) \mid X_1, \ldots, X_n] \underset{\text{Jensen's Ineq}}{\geq} g(E[X_{n+1} \mid X_1, \ldots, X_n])$

$= g(X_n) = Z_n.$

b) Note ~~that~~ g is concave if $h = -g$ is convex. By a) and 8)

$E(-Z_{n+1} \mid -Z_1, \ldots, -Z_n) = E[-Z_{n+1} \mid X_1, \ldots, X_n]$

$= E(-g(X_{n+1}) \mid X_1, \ldots, X_n) \geq -g(X_n).$

$\therefore E(Z_{n+1} \mid Z_1, \ldots, Z_n) \leq g(X_n) = Z_n.$

10) For the martingale CLT, the X_i need not be independent, and $\{T_n\}$ can be shown to be a martingale with $E(T_n) = 0$ and $V(T_n) = E(T_n^2) = a_n^2$.

Thus $Z_n = T_n / a_n$ is the zscore of T_n.

11) <u>Martingale CLT</u>: Let $X_1, X_2, \ldots$ be a sequence of RVs with $E(X_k) = 0$, $V(X_k) = E(X_k^2) = \sigma_k^2 < \infty$, and $E(X_k \mid X_1, \ldots, X_{k-1}) = 0$ with $X_0 = 0$.

Let $T_n = \sum_{k=1}^{n} X_k$, $\Delta_n^2 = \sum_{k=1}^{n} \sigma_k^2$,

$v_k = E(X_k^2 \mid X_1, \ldots, X_{k-1})$, and

$w_n^2 = \sum_{k=1}^{n} v_k^2$. If

a) $\frac{w_n^2}{\Delta_n^2} \xrightarrow{P} 1$ as $n \to \infty$,

b) Lindeberg's condition: for any $\varepsilon > 0$

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{E\left(X_k^2 \, I\left[|X_k| \geq \varepsilon \Delta_n\right]\right)}{\Delta_n^2} = 0,$$

then $Z_n = \frac{T_n}{\Delta_n} \xrightarrow{D} N(0,1)$.

12) Show T_n is a martingale.

Soln 0) Take $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

M1) $T_n = \sum_{k=1}^{n} X_k$ is a function of $X_1, \ldots, X_n$

and hence measurable $\mathcal{F}_n$

M2) holds since $E(T_n) = 0 \in \mathbb{R}$ $\forall n$.

For M3) (note that $E(T_n | X_1, \ldots, X_n)$
$= E(T_{n+1} | T_1, \ldots, T_n)$ since if
$X_1, \ldots, X_n$ are known, then $T_1, \ldots, T_n$ are known, and if $T_1, \ldots, T_n$ are known, then $T_1 = X_1$ and $T_k - T_{k-1} = X_k$ are known for $k = 2, \ldots, n$.) ← (not really needed)

Thus $E[T_{n+1} | X_1, \ldots, X_n] =$

$$E[T_n + X_{n+1} | X_1, \ldots, X_n] = T_n + \underbrace{E(X_{n+1} | X_1, \ldots, X_n)}_{0 \text{ by assump}}$$