P.16

e.g.) A field is a π-system

and a σ-field is a λ-system.

A λ-system need not be a σ-field.

P37 8) Dynkin's π-λ Theorem!

If Π is a π-system and Λ is a λ-system, then $\Pi \subseteq \Lambda$

$\Rightarrow \sigma(\Pi) \subseteq \Lambda$.

proof see p37-38.

P38 9) Th: Suppose P_1 and P_2 are probability measures on $\sigma(\Pi)$ where Π is a π-system. If P_1 and P_2 agree on Π, then they agree on $\sigma(\Pi)$.

proof) Let Λ be the class of sets A in $\sigma(\Pi)$ such that $P_1(A) = P_2(A)$.

$\Omega \in \Lambda$

If $A \in \Lambda$ then $P_1(A^c) = 1 - P_1(A)$

$= 1 - P_2(A) = P_2(A^c)$, $\therefore A^c \in \Lambda$.

If A_n are disjoint sets in Λ, then

$$P_1\left(\bigcup_n A_n\right) = \sum_n P_1(A_n) = \sum_n P_2(A_n)$$

$$= P_2\left(\bigcup_n A_n\right).$$ Thus $\bigcup_n A_n \in \Lambda$.

Hence Λ is a λ-system.

Since $\Pi \subseteq \Lambda$, by the Π-λ theorem, $\sigma(\Pi) \subseteq \Lambda$.

$\square$

§14 Distribution Functions

p184 1] A RV is a measurable real function X on a prob space $(\Omega, \mathcal{F}, P)$. The distribution or law of RV X = induced measure from X on $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$ is the prob measure

$$\mu(A) = P_X(A) = P(X \in A),\ A \in \mathbb{B}(\mathbb{R})$$

$$= P[\omega : X(\omega) \in A].$$

(cumulative)

2) Know the distribution function of X is $F(x) = \mu((-\infty, x]) = P[X \le x]$ for $x \in \mathbb{R}$.

3) $F(x-) = P(X < x)$.

$P(X = x) = F(x) - F(x-)$.

$F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

$F(\infty) = \lim_{x \to \infty} F(x) = 1$

$F(x)$ is a non decreasing function of x: if $x_1 < x_2$, then $F(x_1) \le F(x_2)$.

$F(x)$ is right continuous:

$\lim_{x \downarrow x_0} F(x) = F(x_0)$ for any $x_0 \in \mathbb{R}$.

$F(x)$ can have at most countably infinite points of discontinuity.

$P(a < X \le b) = F(b) - F(a)$

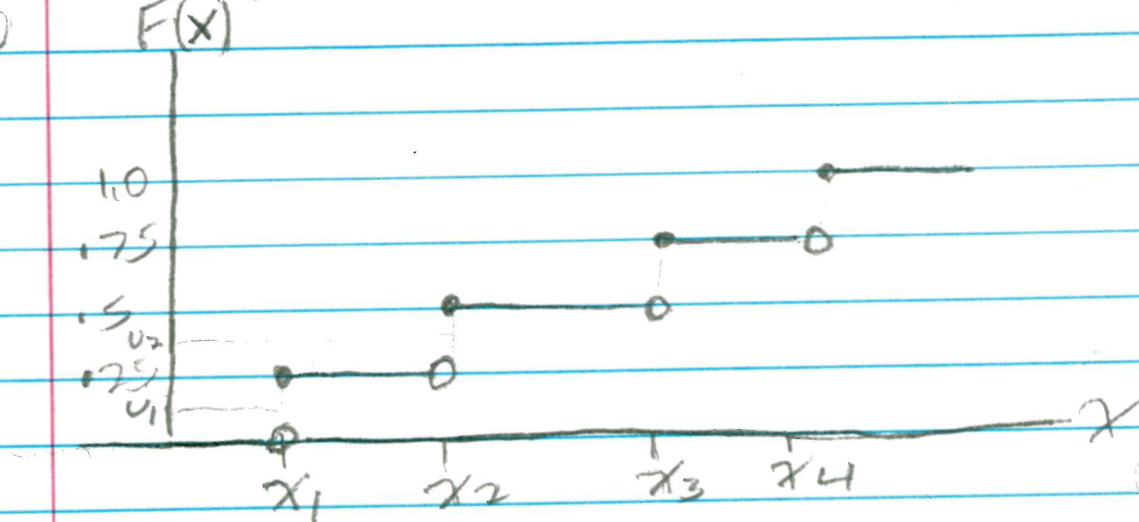
4) The quantile function

$Q(u) = \inf\{x : u \le F(x)\}, \quad 0 < u < 1$

5] If F^{-1} exists,

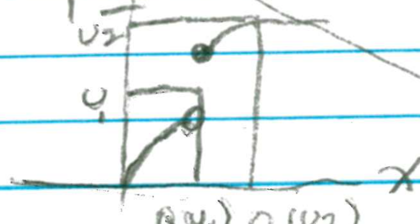
$Q(u) = F^{-1}(u)$. $Q(u) \le x$ iff $u \le F(x)$

ex)



If $F(x) = \begin{cases} 0 & x < x_1 \\ 0.25 & x_1 \le x < x_2 \\ 0.5 & x_2 \le x < x_3 \\ 0.75 & x_3 \le x < x_4 \\ 1 & x \ge x_4 \end{cases}$, then

$Q(u) = \begin{cases} x_1 & 0 < u \le 0.25 \\ x_2 & 0.25 < u \le 0.5 \\ x_3 & 0.5 < u \le 0.75 \\ x_4 & 0.75 < u < 1 \end{cases}$



PM 18

RVs and

Random Vectors §13, 20, 4

p184 1) Fix $(\Omega, \mathcal{F}, P)$, A function

$\underset{\sim}{X} : \Omega \to \mathbb{R}^k$ is a $1 \times k$

random vector if

$\underset{\sim}{X}^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{B}(\mathbb{R}^k)$.

iff $\underset{\sim}{X} : \Omega \to \mathbb{R}^k$ is a

measurable function.

p265 2) The distribution function

of a random vector $\underset{\sim}{X}$ is

$F(\underset{\sim}{x}) = P(X_1 \leq x_1, \ldots, X_k \leq x_k)$.

3) A σ field $\sigma(\underset{\sim}{X})$ is

the smallest σ field with respect to which $\underset{\sim}{X}$ is measurable

A σ field $\sigma(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_j)$ generated

$1 \times k$

by a collection of random vectors is the smallest σ field with respect to each one is measurable

$\underline{X}^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathbb{B}(\mathbb{R}^k)$

$\underset{\sim}{X}(\omega) = (X_1(\omega), \ldots, X_k(\omega))$.

4) Random variables $X_1, \ldots, X_k$ are independent if

$\sigma(X_1), \ldots, \sigma(X_k)$ are ind

$(A_i \in \sigma(X_i) \Rightarrow A_1, \ldots, A_k$ are ind$)$

$X_1, \ldots, X_k$ are ind iff

$P(X_1 \in B_1, \ldots, X_k \in B_k) =$

$P(X_1 \in B_1) \cdots P(X_k \in B_k)$ for

any sets $B_1, \ldots, B_k \in \mathbb{B}(\mathbb{R})$.

Know

$X_1, \ldots, X_k$ are ind if

$$F(x_1, \ldots, x_k) = F_{X_1}(x_1) \cdots F_{X_k}(x_k)$$

for any real $x_1, \ldots, x_k$,

Note: replace x_i by ∞

and the above equation still holds with x_i omitted. Hence the above equation holds for any subcollection $x_{i_1}, \ldots, x_{i_j}$:

$$F(x_{i_1}, \ldots, x_{i_j}) = F_{X_{i_1}}(x_{i_1}) \cdots F_{X_{i_j}}(x_{i_j})$$

if $X_1, \ldots, X_k$ are ind.

So showing independence of k RVs is much easier than showing independence of K events $A_1, \ldots, A_k$.

* 5] Th: Fix $(\Omega, \mathcal{F}, P)$. For a RV X, the set function

$$\underbrace{P_X}_{\text{set function on } \mathbb{B}(\mathbb{R})}(B) = \underbrace{P}_{\text{set function on } \mathcal{F}}[X^{-1}(B)], \quad B \in \mathbb{B}(\mathbb{R})$$

is a probability measure on

$(\mathbb{R}, \mathbb{B}(\mathbb{R}))$.

Fnd21

Proof: P1) Let $B \in \mathbb{B}(\mathbb{R})$. Then

$P_X(B) = P(X^{-1}(B))$. so

$0 \leq P_X(B) \leq 1$.

P2) $P_X(\mathbb{R}) = P(X^{-1}(\mathbb{R}))$

$= P(\Omega) = 1$

$P_X(\phi) = P(X^{-1}(\phi)) =$

$P(\{\omega : X(\omega) \in \phi\}) = P(\phi) = 0$.

or ϕ has no elements

$X(\omega) \in \mathbb{R} \ \forall \omega \in \Omega$

P3) Let $\{B_i\}_{i=1}^{\infty}$ be disjoint $\mathbb{B}(\mathbb{R})$ sets.

Then $P_X\left(\bigcup_{i=1}^{\infty} B_i\right) = P\left[X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right]$

$= P\left[\bigcup_{i=1}^{\infty} X^{-1}(B_i)\right]$.

HW 3

PM 20

Now B_i disjoint and X a function $\Rightarrow$ $X^{-1}(B_i) =$

$\{\omega : X(\omega) \in B_i\}$ are disjoint

$\therefore$ $P_X\left(\bigcup_{i=1}^{\infty} B_i\right) =$

$\sum_{i=1}^{\infty} P\left(X^{-1}(B_i)\right) = \sum_{i=1}^{\infty} P_X(B_i)$

$\square$

Note $X^{-1}(B_i) \in \mathcal{F}$ since X is a RV.

The proof for $X = \underline{X}$ is the same except $P_X(\mathbb{R}^k) = P(\Omega) = 1$, $\mathbb{B}(\mathbb{R}) \to \mathbb{B}(\mathbb{R}^k)$

6) a) The distribution of X is

$P_X(B) = P(X^{-1}(B))$, $B \in \mathbb{B}(\mathbb{R})$

b) The distribution function of X. is $F(t) = F_X(t) = P_X((-\infty, t])$.

P_X is a set function

F_X is a function as in calculus.

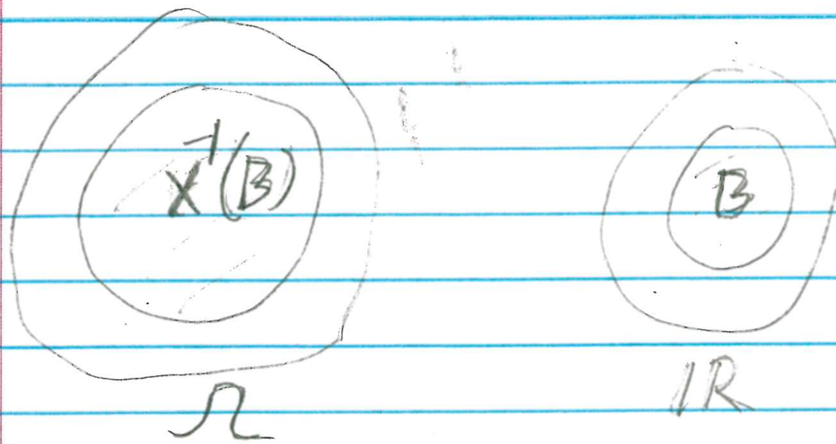
$P_X((-\infty, t]) = P\left(X^{-1}((-\infty, t])\right) =$

$P\left(\{\omega : X(\omega) \in (-\infty, t]\}\right) = P(X \le t)$

and $(-\infty, t] \in \mathbb{B}(\mathbb{R})$.

20.5

$X : \Omega \to \mathbb{R}$



Thm $\sigma(X) =$ the collection

$\{ X^{-1}(B) : B \in \mathbb{B}(\mathbb{R}) \}$

is a σ-field.

Proof) $\sigma 1)$ $X^{-1}(\mathbb{R}) = \Omega \in \sigma(X)$

$\sigma 2)$ Let $A \in \sigma(X)$. Then

$A = X^{-1}(B)$ for some $B \in \mathbb{B}(\mathbb{R})$.

Claim $[X^{-1}(B)]^c = X^{-1}(B^c)$

pf: $B \cup B^c = \mathbb{R}$

$X^{-1}(B \cup B^c) = X^{-1}(\mathbb{R}) = \Omega$

$\therefore X^{-1}(B) \cup X^{-1}(B^c) = \Omega$

$X^{-1}(B)$ and $X^{-1}(B^c)$ are disjoint

Since B and B^c are disjoint.

$\therefore [X^{-1}(B)]^c = X^{-1}(B^c)$

$\therefore A^c = X^{-1}(B^c) \in \sigma(X)$.

$\sigma 3)$ $A, B \in \sigma(X) \Rightarrow A = X^{-1}(C), B = X^{-1}(D)$, $A \cap B = X^{-1}(C \cap D) \in \sigma(X)$.

$\sigma 4)$ Let $\{A_i\}_{i=1}^{\infty} \in \sigma(X)$.

Then $A_i = X^{-1}(B_i)$ for some

$B_i \in \mathbb{B}(\mathbb{R})$. $\therefore$

$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} X^{-1}(B_i) =$

↑HW 3

$X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) \in \sigma(X)$

$\square$

8) Now let probability measure

(P was Px) $P : \mathbb{B}(\mathbb{R}) \to [0,1]$. Let the

distribution function of P be

$F_P : \mathbb{R} \to [0,1]$ be defined

by $F_P(t) = P((-\infty, t])$.

Note: $\mathbb{B}(\mathbb{R}) = \sigma(A)$ where

A = class of $(a,b]$: $a<b$, $a,b \in \mathbb{R}$

9) Th: $F_P = F_{P'}$ iff $P = P'$.

(If real functions $F_P = F_{P'}$ then the set functions $P = P'$. There is a 1-1 correspondence between F_P and P.)

Proof: $P' = P \Rightarrow F_P = F_{P'}$. Suppose $F_P = F_{P'}$.

Let $\Pi = \{(-\infty, b], b \in \mathbb{R}\}$.

Then $(-\infty, b_1] \cap (-\infty, b_2]$

$= (-\infty, \min(b_1, b_2)] \in \Pi$ and

Π is a π-system and

$\mathbb{B}(\mathbb{R}) = \sigma(\Pi)$.

$P((-\infty, b]) = P'((-\infty, b])$
since $F_P = F_{P'}$.

PM #2

Hence $P = P'$ on π and by P3 [9], $P = P'$ on $\sigma(\pi) = \mathbb{B}(\mathbb{R})$.

((Show $\mathbb{B}(\mathbb{R}) = \sigma(\pi)$.)

Let $\pi' = \{(a,b] : a < b,\ a,b \in \mathbb{R}\} \cup \phi$

Then $\mathbb{B}(\mathbb{R}) = \sigma(\pi')$ and π' is a π-system. Now

$(a_1,b_1] \cap (a_2,b_2] = \phi$ or $(\max a_1,a_2, \min b_1,b_2]$

$P((a,b]) = P[(-\infty,b] - (-\infty,a]]$

$= P((-\infty,b]) - P((-\infty,a]) =$

$P'((-\infty,b]) - P'((-\infty,a])$

$= P'((a,b])$. $\therefore$ P and P' agree on π' and hence on

$\sigma(\pi') = \mathbb{B}(\mathbb{R})$.)

$\square$

22.5

10) Th A function $F: \mathbb{R} \to [0,1]$

is a distribution function
(of a RV X or of set function P_X)
if

df1) F is nondecreasing:
$x_1 < x_2 \Rightarrow F(x_1) \le F(x_2)$

df2) F is right continuous:
$\lim_{h \downarrow 0} F(x+h) = F(x) \quad \forall x \in \mathbb{R}$

df3) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

df4) $F(\infty) = \lim_{x \to \infty} F(x) = 1$.

proof: $F(x) = P((-\infty, x])$.

df1) If $x_1 < x_2$ then
$(-\infty, x_1] \subseteq (-\infty, x_2]$.

So $F(x_1) = P((-\infty, x_1]) \le P((-\infty, x_2]) = F(x_2)$.

df2) As $h \downarrow 0$, $(-\infty, x+h] \downarrow (-\infty, x]$,

$\therefore F(x+h) \downarrow F(x)$.

df3) $(-\infty, -n] \downarrow \phi$
$F(-n) \downarrow 0$

$$\lim_{n\to\infty} F(-n) = \lim_{x\to-\infty} F(x) = 0$$

∂f 4) $(-\infty, n] \uparrow \mathbb{R}$

$F(n) \uparrow P(\mathbb{R}) = 1$

$\square$

Note: technically

$P((-\infty, x+h]) \downarrow P((-\infty, x])$ is a countable limit, but

$(-\infty, x+h] \downarrow (-\infty, x]$ regardless of how $h \downarrow 0$ (eg countable: $h = \frac{1}{n}$ rationals or uncountable: irrationals)

and $(-\infty, x]$ and $(-\infty, x+h]$ are Borel sets.

11] A measure μ is not continuous if $\mu(\Omega) = \infty$.

ex] $X = c$ $(X(\omega) \equiv c \ \forall \omega)$ is a RV

proof: $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathbb{B}(\mathbb{R}), P_X)$ is a RV if $X^{-1}(B) \in \mathcal{F} \ \forall B \in \mathbb{B}(\mathbb{R})$.

23.5

Let $B \in \mathbb{B}(\mathbb{R})$. Then

$$X^{-1}(B) = \{\omega : X(\omega) = c \in B\}$$

$$= \{\omega : c \in B\} = \begin{cases} \emptyset & c \notin B \\ \Omega & c \in B \end{cases}$$

$\therefore X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathbb{B}(\mathbb{R})$

Recall $\sigma(X)$ = the collection

of $\{X^{-1}(B) : B \in \mathbb{B}(\mathbb{R})\}$.

If $X \equiv c$, then

$\sigma(X) = \{\emptyset, \Omega\}$, the smallest σ-field.

The simpler the RV X, the simpler $\sigma(X)$ is.

ex) Claim: $X = I_A$ is a RV iff $A \in \mathcal{F}$.

(iff A is an event)

$$X(\omega) = I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$