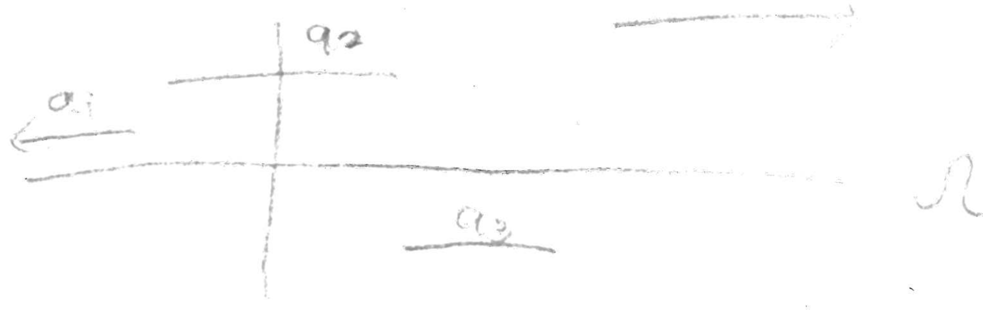
$X^{-1}(B) = \{\omega : \underbrace{I_A(\omega)}_{0 \text{ or } 1} \in B\}$

$$X^{-1}(B) = \begin{cases} \phi & \text{if } 0 \notin B \text{ and } 1 \notin B \\ A^c & \text{if } 0 \in B \text{ and } 1 \notin B \\ A & \text{if } 0 \notin B \text{ and } 1 \in B \\ \Omega & \text{if } 0 \in B \text{ and } 1 \in B \end{cases}$$

(If $0 \in B$ and $1 \notin B$, then $X^{-1}(B) = \{\omega : I_A(\omega) \in B\} = \{\omega : I_A(\omega) = 0\} = A^c$.)

Note: $\sigma(I_A) = \{\phi, A, A^c, \Omega\}$ and $A \in \mathcal{F}$ iff $A^c \in \mathcal{F}$.

ex) Suppose X is a function that takes on several values.



eg $X(\omega) = a_1 I_{A_1}(\omega) + a_2 I_{A_2}(\omega) + a_3 I_{A_3}(\omega) + a_4 I_{A_4}(\omega)$

where $A_i = \{\omega : X(\omega) = a_i\}$.

Such a stepfunction is a linear combination of indicators.

p185 12] A function is <u>simple</u> if it is a linear combination of indicators

$$f = \sum_{i=1}^{K} x_i I_{A_i}$$ for some positive

integer K (so finite range).

13] A simple function is a RV iff each $A_i \in \mathcal{F}$.

End Exam 1 material

Begin Exam 2 material

Integration and Expected Value

§5, 15–18; 21
ch3 ch4

§3

P63 1] Fix $(\Omega, \mathcal{F}, P)$. A <u>simple random variable</u> is a function $X: \Omega \to \mathbb{R}$ such that the range of X is finite and $\{X=x\} = \{\omega : X(\omega)=x\} \in \mathcal{F}$ $\forall x \in \mathbb{R}$.
(X is a discrete RV with finite support.)

ex] $X = \sum_{i=1}^{n} x_i I_{A_i}$, $A_i \in \mathcal{F}$ is a simple RV.

ex] A_n $n \geq 1$ disjoint, $X = \sum_{i=1}^{\infty} x_i I_{A_i}$, where $x_i \neq x_j$ for $i \neq j$, is <u>not</u> a simple RV since X has infinite range.

2] Suppose events $A_1, \ldots, A_n$ are disjoint and $\bigcup_{i=1}^{n} A_i = \Omega$. Let $X = \sum_{i=1}^{n} x_i I_{A_i}$.

The <u>expected value</u> of X =

I use for proofs

$$E[X] = \sum_{i=1}^{n} x_i P(A_i) = \sum_{x} x P(X=x)$$

which is a finite sum since X is a simple RV.

3) Existence and uniquess of 2).

Existence: Suppose X takes on distinct values $x_1, \ldots, x_m$ (m need not equal n).

Let $B_i = \{X = x_i\} = \{w : X(w) = x_i\}$ for $i = 1, \ldots, m$. Then the B_i are disjoint and $\bigcup_{i=1}^{m} B_i = \Omega$.

Thus $X = \sum_{i=1}^{m} x_i I_{B_i}$ and $E(X) = \sum_{i=1}^{m} x_i P(B_i)$

$= \sum_{i=1}^{m} x_i P(X = x_i)$.

Uniqueness: $\sum_{i=1}^{n} x_i P(A_i) =$

$$\sum_{x} \underbrace{\sum_{i: x_i = x} x_i P(A_i)}_{x_i = x \text{ in this sum}} = \sum_{x} x\, P\Big(\underbrace{\bigcup_{i: x_i = x} A_i}_{\{X = x\}}\Big)$$

(finite sum)

$$= \sum_{x} x\, P(X = x),$$

Note: Although many partitions A_i's, exist, they all yield the same value of $E(X)$.

(Existence in 3] shows X can be written as in 2])

4] Let X_n, X and Y be simple RVS.

a] $-\infty < E[X] < \infty$.

b] $E[aX + bY] = aE(X) + bE(Y)$ linearity

c] If $X \le Y$, then $E(X) \le E(Y)$

d] If $\{X_n\}$ is uniformly bounded and $X = \lim_n X_n$ on a set of prob 1, then $E(X) = \lim_n E(X_n)$

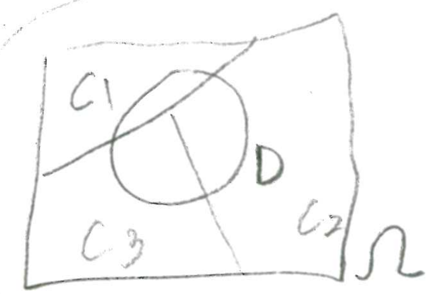
(special case of LDCT)

e) $E[t(X)] = \sum_{x} t(x) P(X=x)$. t a real valued function

f) If X is nonnegative, $X \geq 0$,

then $E[X] = \sum_{i} P(X > x_i) = \int_0^\infty P(X > x) dx$

$= \int_0^\infty \underbrace{[1 - F(x)]}_{\text{survival function } S(x)} dx$.



g) If $X \perp\!\!\!\perp Y$, $E[XY] = E[X] E[Y]$.

some proofs: a) $E[X] = \sum_{x} \underbrace{x}_{\text{bounded}} \underbrace{P(X=x)}_{\in [0,1]}$. The result

holds since X has finite range $x_1, \ldots, x_m$.

$\min(x_i) \leq E[X] \leq \max(x_i)$.

(c) $E(X) = \sum_{i} x_i P(A_i)$ and $E(Y) = \sum_{j} y_j P(B_j)$.

skip

Since $X \leq Y$ $\forall \omega$, $x_i \leq y_j$ if $\underbrace{A_i \cap B_j}_{\text{disjoint}} \neq \emptyset$

$\therefore E[X] = \sum_{i} x_i P(A_i) \overset{\text{venn diagram}}{=} \sum_{i} \sum_{j} x_i P(A_i \cap B_j)$

$\leq \sum_{i} \sum_{j} y_j P(A_i \cap B_j) = \sum_{j} y_j P(B_j) = E[Y]$)

$= \sum_{j} \sum_{i} y_j P(A_i \cap B_j)$

c) Let $W = Y - X \geq 0$

$$E(W) = \underbrace{\sum w}_{\geq 0} \underbrace{p(W=w)}_{\geq 0} \geq 0$$

$$\therefore 0 \leq E[Y-X] = E[Y] - E(X)$$ (by b)

or $E(X) \leq E(Y)$.

e) If $X = \sum_{i=1}^{n} x_i I_{A_i}$, then

$$\underbrace{t(X)}_{W} = \underbrace{\sum_{i=1}^{n} t(x_i) I_{A_i}}_{W \text{ as a simple RV}}$$

$$\therefore E[t(X)] = E[W] = \sum w\, P(W=w)$$

$$= \sum_{i=1}^{n} t(x_i) P(A_i).$$

g) $XY = \sum_i x_i I_{A_i} \sum_j y_j I_{B_j} =$

$\sum_i \sum_j x_i y_j \underline{I_{A_i \cap B_j}}$ is a SRV

$(I_{A_i} I_{B_j} \in \{0,1\}$ and $I_{A_i} I_{B_j} = 1$ iff $I_{A_i} = I_{B_j} = 1$ iff $I_{A_i \cap B_j} = 1)$

So $E[XY] = \sum_i \sum_j x_i y_j P(A_i \cap B_j) \underset{\text{ind}}{=}$

$$\sum_i \sum_j x_i \, y_j \, P(A_i) \, P(B_j) =$$

$$\sum_i x_i \, P(A_i) \sum_j y_j \, P(B_j) = E(X)E(Y)$$

Note: $\underbrace{I_{A_1} I_{A_2} \cdots I_{A_m}}_{\in\{0,1\} \text{ and } 1 \text{ iff } I_{\bigcap_{i=1}^m A_i} = 1} = I_{A_1 \cap A_2 \cap \cdots \cap A_m} = I_{\bigcap_{i=1}^m A_i}$

b) Let $X = \sum_i x_i I_{A_i}$ and $Y = \sum_j y_j I_{B_j}$.

Then $aX + bY = a x_i + b y_j$ for $\omega \in \underbrace{A_i \cap B_j}_{\text{partition } \Omega}$

So $aX + bY = \sum_i \sum_j (a x_i + b y_j) I_{A_i \cap B_j}$ is a SRV

with $E[aX + bY] = \sum_i \sum_j (a x_i + b y_j) P(A_i \cap B_j) =$

$$\sum_i a x_i \sum_j P(A_i \cap B_j) + \sum_j b y_j \sum_i P(A_i \cap B_j)$$

$$= a \sum_i x_i P(A_i) + b \sum_j y_j P(B_j) = aE(X) + bE(Y)$$

$\square$