5) Th: Given a nonnegative RV X

$\exists$ simple RVs X_n which are

nonnegative $\ni$ $X_n \uparrow X$ $(X_n(\omega) \uparrow X(\omega)\ \forall \omega)$

Analogy: Take step functions, increase

them to get Riemann integrability of

a function.

6) $\exists \{X_n\}$ SRVS $\ni$

a) $X_n(\omega) \uparrow X(\omega)$ if $X(\omega) \geq 0$

b) $X_n(\omega) \downarrow X(\omega)$ if $X(\omega) \leq 0$

ϕ 3.15

7) For the theory of integration assume the function f in the integrand is measurable (measurable $\mathcal{F}$ where $f: \Omega \to \mathbb{R}$ Borel measurable) and $(\Omega, \mathcal{F}, \mu)$ is a measure space.

8) Def: Let f be a nonnegative function

$f: \Omega \to [0, \infty]$.

∞ allowed

The <u>integral</u> $\int f \, d\mu = \int_\Omega f(\omega)\, \mu(d\omega)$

$$= \sup_{\{A_i\}} \sum_i \left(\inf_{\omega \in A_i} f(\omega)\right) \mu(A_i)$$

where $\{A_i\}$ is a finite $\mathcal{F}$ decomposition.

("

means that $A_i \in \mathcal{F}$, $\Omega = A_1 \cup \dots \cup A_n$
for some n, A_i's are disjoint)

9} conventions for integration: a) $A_i = \phi \Rightarrow$ the inf $= \infty$

b) $x(\infty) = \infty$, $x > 0$

c) $0(\infty) = 0$.

(compare $E(X)$ for X a SRV)

10} Th: Let $f \geq 0$.

i) If $f(\omega) = \sum_{j=1}^{m} x_j I_{B_j}(\omega)$, $x_j \geq 0$,

$\{B_j\}$ an $\mathcal{F}$ decomp of Ω,

then $\int f \, d\mu = \sum_{j=1}^{m} x_j \mu(B_j)$.

ii) $0 \le f \le g \Rightarrow \int f\,d\mu \le \int g\,d\mu$

iii) $0 \le f_n \uparrow f \Rightarrow \int f_n\,d\mu \uparrow \int f\,d\mu$

iv) $a, b \ge 0,\ f, g \ge 0 \Rightarrow$

$$\int (af + bg)\,d\mu = a\int f\,d\mu + b\int g\,d\mu$$

some proofs: i) want to show

$$\sum_i \left(\inf_{A_i} f(\omega)\right)\mu(A_i) \le \underbrace{\sum_{j=1}^m x_j\,\mu(B_j)}_{\text{want this to be the SUP of LHS sums}}$$

$$= \sum_{j=1}^m \inf_{\omega \in B_j}(f(\omega))\,\mu(B_j).$$

← shows middle term has the form of LHS

Now $\sum_{i=1}^n \left(\inf_{A_i} f(\omega)\right)\mu(A_i) \underset{\uparrow}{=}$

B_j's form a decomp

$$\sum_{i=1}^n \left(\inf_{A_i} f(\omega)\right)\sum_{j=1}^m \mu(A_i \cap B_j)$$

$$= \sum_{i=1}^n \sum_{j=1}^m \left(\inf_{A_i} f(\omega)\right)\mu(A_i \cap B_j) \underset{\uparrow}{\le}$$

if $C \subseteq D$ then $\inf_D f(\omega) \le \inf_C f(\omega)$

$$\sum_{i=1}^{n} \sum_{j:\, A_i \cap B_j \neq \emptyset} \underbrace{\left(\inf_{A_i \cap B_j} f(\omega) \right)}_{x_j} \mu(A_i \cap B_j)$$

$$= \sum_{j=1}^{m} x_j \sum_{i=1}^{n} \mu(A_i \cap B_j) =$$

add terms $= 0$

$$\sum_{j=1}^{m} x_j \, \mu\Big(\underbrace{\Big(\bigcup_{i=1}^{n} A_i \Big)}_{\Omega} \cap B_j \Big) =$$

$$\sum_{j=1}^{m} x_j \, \mu(B_j).$$

ii) Since $f \leq g \Rightarrow \inf_A f(\omega) \leq \inf_A g(\omega)$,

$$\sum_i \left(\inf_{A_i} f(\omega) \right) \mu(A_i) \leq \sum_i \left(\inf_{A_i} g(\omega) \right) \mu(A_i)$$

and the result follows. $\square$

II) Now let $f : \Omega \to [-\infty, \infty]$.
allowed

the positive part

$$f^+(\omega) = \begin{cases} f(\omega) & \text{if } 0 \le f(\omega) \le \infty \\ 0 & \text{if } -\infty \le f(\omega) \le 0 \end{cases}$$

$$= f(\omega)\, I(0 \le f(\omega) \le \infty)$$

$f^+ = f\, I(f \ge 0)$

and the negative part

$$f^-(\omega) = \begin{cases} -f(\omega) & \text{if } -\infty \le f(\omega) \le 0 \\ 0 & \text{if } 0 \le f(\omega) \le \infty \end{cases}$$

$= -f(\omega)\, I[-\infty \le f(\omega) \le 0]$ $f^- = -f\, I(f \le 0)$

Then f^- and f^+ are measurable and nonnegative with $f = f^+ - f^-$ (assume not $\infty - \infty$)

$|f| = f^+ + f^-$

p203

12) * Def $\int f\, d\mu = \int f^+ d\mu - \int f^- d\mu$.

This integral is defined unless it involves "$\infty - \infty$". We say f is integrable ^wrt^ if both $\int f^+ d\mu$ and $\int f^- d\mu$ are finite.

30.5

13) "Almost everywhere" (ae) means the property holds for ω outside a set of measure 0
(So on a set A where $\mu(A^c)=0$)

14) ~~Th~~ Suppose f and g are nonnegative.

i) If $f=0$ ae, then $\int f\,d\mu=0$.

ii) If $\mu(\{\omega : f(\omega)>0\})>0$, then

$\int f\,d\mu>0$.

iii) If $\int f\,d\mu<\infty$, then $f<\infty$ ae.

iv) If $f\leq g$ ae, then $\int f\,d\mu\leq\int g\,d\mu$.

v) If $f=g$ ae, then $\int f\,d\mu=\int g\,d\mu$.

Some proofs: i) If there is $\omega\in A_i$ $\ni f(\omega)=0$, then $\inf_{\omega\in A_i} f(\omega)=0$, otherwise $\mu(A_i)=0$. Hence the integral in 8)
$=0$.

v) $f = g$ ae means $f \leq g$ ae and $g \leq f$ ae. Hence the result follows by iv).

§16

15] Th i) f is integrable iff $\int |f| d\mu < \infty$.

ii) If f and g are integrable and $f \leq g$ ae, then $\int f d\mu \leq \int g d\mu$ (monotonicity)

iii) linearity: If f and g are integrable and $a, b \in \mathbb{R}$, then $af + bg$ is integrable with $\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$

iv) monotone convergence th (MCT): If $0 \leq f_n \uparrow f$ ae, then $\int f_n d\mu \uparrow \int f d\mu$

v) <u>Fatou's lemma</u>: For nonnegative f_n,

$$\int \liminf_n f_n \, d\mu \le \liminf_n \int f_n \, d\mu.$$

(LDCT)

vi) <u>Lebesgue's Dominated Convergence Th</u>:

If $|f_n| \le g$ ae where g is integrable, and if $f_n \to f$ ae, then f and f_n are integrable and $\int f_n \, d\mu \to \int f \, d\mu$.

vii) Bounded convergence th (BCT):

If $\mu(\Omega) < \infty$ and the f_n are uniformly bounded, then $f_n \to f$ ae $\Rightarrow$

$\int f_n \, d\mu \to \int f \, d\mu$.

viii) If $f_n \ge 0$, then $\int \sum_{n=1}^{\infty} f_n \, d\mu$

$= \sum_{n=1}^{\infty} \int f_n \, d\mu$

ix) If $\sum_{n=1}^{\infty} \int |f_n| \, d\mu < \infty$ then

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

x) $\left| \int f \, d\mu - \int g \, d\mu \right| \leq \int |f-g| \, d\mu$

some proofs} i) If f is integrable then

$\int |f| \, d\mu = \int f^+ d\mu + \int f^- d\mu < \infty$. If $\int |f| \, d\mu < \infty$,

then $\int |f| \, d\mu = \underbrace{\int f^+ d\mu}_{\geq 0} + \underbrace{\int f^- d\mu}_{\geq 0} < \infty$. $\therefore$

$\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$ and f is integrable

ii) case a) Let f and g be nonnegative $\ni f \leq g$ ae

Then $\int f \, d\mu \leq \int g \, d\mu$ by 14} iv).

case b) For general f and g, if $f \leq g$ ae

then $f = f^+ - f^- \leq g^+ - g^-$ ae.

← ~~sets where f^+ and f^+ are~~ ($f = f^+$ or f^-)

$\therefore$ $f^+ \leq g^+$ ae and $\underbrace{-f^- \leq -g^-}_{f^- \geq g^-}$ ae

Thus $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu \leq \int g^+ d\mu - \int g^- d\mu$

$= \int g \, d\mu$ by case a).

32.5

iv) If $0 \le f_n \uparrow f$ on a set A with $\mu(A^c) = 0$, then $0 \le f_n I_A \uparrow f I_A$ everywhere.

$\therefore \int f_n I_A \, d\mu \uparrow \int f I_A \, d\mu$ by [10] iii),

Since $f_n = f_n I_A$ ae and $f = f I_A$ ae,

$$\int f_n \, d\mu = \int f_n I_A \, d\mu \uparrow \int f I_A \, d\mu = \int f \, d\mu$$

by [14] v).

v) $\liminf_n f_n = \lim_{n \to \infty} \left(\inf_{k \ge n} f_k \right) = \lim_{n \to \infty} g_n$

where $g_n = \inf_{k \ge n} f_k$. Note $0 \underset{\substack{\uparrow \\ f_n \ge 0}}{\le} g_n \uparrow \liminf_n f_n$.

$$\therefore \int \liminf_n f_n \, d\mu = \int \lim_n g_n \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu$$

by MCT, if $0 \le g_n \uparrow g$ ae, then $\lim_{n \to \infty} \int g_n \, d\mu = \int g \, d\mu = \int \lim_{n \to \infty} g_n \, d\mu$

$$= \liminf_n \int g_n \, d\mu \le \liminf_n \int f_n \, d\mu$$

Since limit exists

$g_n = \inf_{k \ge n} f_k$, in particular, $g_n \le f_n$

vii) Let $|f_n| \le K$ $\forall n$ and $\forall \omega$. Then

$$|f_n(\omega)| \le K\, I_\Omega(\omega) = g(\omega).$$

$$\int g\, d\mu = K \int I_\Omega\, d\mu = K\, \mu(\Omega) < \infty.$$

$\therefore$ g is integrable.

Then LDCT $\Rightarrow$ f and f_n are integrable and $\int f_n\, d\mu \to \int f\, d\mu$.

viii) $0 \le g_n = \sum_{m=1}^{n} f_m \uparrow g = \sum_{n=1}^{\infty} f_n$.

MCT $\Rightarrow$ $\int g_n\, d\mu \uparrow \int g\, d\mu$ where

$$\int g_n\, d\mu = \int \sum_{m=1}^{n} f_m\, d\mu \underset{\substack{\uparrow \\ \text{linearity}}}{=} \sum_{m=1}^{n} \int f_m\, d\mu$$ and

$$\int g\, d\mu = \int \sum_{n=1}^{\infty} f_n\, d\mu$$

$\square$

16) By convention, $\infty - \infty$ is undefined.

17) Common technique: show result is

true for indicators. Extend to simple functions by linearity, and then to nonnegative functions by a monotone passage of the limit.

18] Def: A function $f: \Omega \to [-\infty, \infty]$ is measurable (or $\mathcal{F}$ measurable or Borel measurable) or a measurable function if

i) $f^{-1}(B) \in \mathcal{F}$ $\forall B \in \mathbb{B}(\mathbb{R})$ so $-\infty, \infty \notin B$

ii) $f^{-1}(\{\infty\}) = \{w : f(w) = \infty\} \in \mathcal{F}$ and

iii) $f^{-1}(\{-\infty\}) = \{w : f(w) = -\infty\} \in \mathcal{F}$.

ex] $\int I_A d\mu = \int (1 I_A + 0 I_{A^c}) d\mu$

$= \mu(A) + 0\,\mu(A^c) = \mu(A)$

$\{A, A^c\}$ form a decomp

19] If f is integrable, then $|\int f d\mu| \leq \int |f| d\mu$.

proof: $|\int f d\mu| = |\int f^+ d\mu - \int f^- d\mu| \leq$

tri ineq

$|a-b| = |a + -b| \leq |a| + |-b| = |a| + |b|$ $\quad |a+b| \leq |a| + |b|$

$$|\int f^+ d\mu| + |\int f^- d\mu| \underset{f^-, f^+ \geq 0}{=} \int f^+ d\mu + \int f^- d\mu$$

$$= \int (f^+ + f^-) d\mu = \int |f| d\mu.$$

$\square$

20} consequences

a) linearity $\Rightarrow \int \sum_{n=1}^{K} f_n \, d\mu = \sum_{n=1}^{K} \int f_n \, d\mu$

can interchange finite sum and integral operators

b) MCT, LDCT, BCT give conditions where limit and $\int$ can be interchanged

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu = \int f \, d\mu.$$

c) 15 viii), ix) give conditions when $\sum_{n=1}^{\infty}$ and $\int$ can be interchanged: $\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$

p215 Def:

21} If $A \in \mathcal{F}$, then $\int_A f \, d\mu = \int f I_A \, d\mu$

22} If $\mu(A) = 0$, then $\int_A f \, d\mu = 0$

23] If $\mu : \mathcal{F} \to [0,\infty]$ is a measure and $f \geq 0$, then

a) $\underset{\nu}{\nu}(A) = \int_A f\,d\mu$ is a measure on $\mathcal{F}$.

b) If $\int_\Omega f\,d\mu = 1$, $P(A) = \int_A f\,d\mu$ is a prob measure on $\mathcal{F}$.

Expected Value

24] a) For X, Y SRVs, claimed

$E[a] = a$

monotonicity: $X \leq Y \Rightarrow E(X) \leq E(Y)$

linearity: $E(aX + bY) = aE(X) + bE(Y)$

b) For RV $X \geq 0$, $\exists$ SRVs $X_n \geq 0$

$\ni X_n \uparrow X$ $\quad$ ($X_n(\omega) \uparrow X(\omega)$ $\forall \omega$).

ex] $X_n(\omega) \to X(\omega)$ $\forall\omega$ does not imply

$E[X_n] \to E(X)$ $\quad$ (continuity of $E(X)$ fails)

Let $X_n(\omega) = n\, I_{(0,\frac{1}{n})}$, and $P = U(0,1)$ prob.