Then $E(X_n) = n\,P(0,\tfrac{1}{n}) = n\frac{1}{n} = 1 \quad \forall n$.

$X_n(\omega) \to X(\omega) \equiv 0 \quad \forall\, \omega \in [0,1] = \Omega$.

Thus $E(X_n) \not\to E(X) = 0$.

$E[X_n] \to 1$ Since $E[X_n] \equiv 1 \ \forall n$.

25) Def Let X be a nonnegative RV.

a) $E[X] = \lim_{n\to\infty} E(X_n) \le \infty$ where

X_n are SRVs with $0 \le X_n \uparrow X$.

$\left(E[X] = E\left[\lim_{n\to\infty} X_n\right] = \lim_{n\to\infty} E(X_n) \text{ if } 0 \le X_n \uparrow X,\ X_n \text{ SRVs}\right)$

b) The expectation of X over an event A

is $E[X I_A]$.

26) $0 \le E(X_1) \le E(X_2) \le \dots$

So $\{E[X_n]\}$ is a monotone sequence

and $\lim_{n\to\infty} E[X_n]$ exists in $[0,\infty]$.

27) Uniqueness: If $\{X_n\}$ and $\{X_n'\}$ are

(show $E[X]$ is well defined)

35.5

Sequences of nonnegative SRVs increasing to X as $n \uparrow \infty$, then $\lim_{n\to\infty} E[X_n] = \lim_{n\to\infty} E[X_n']$.

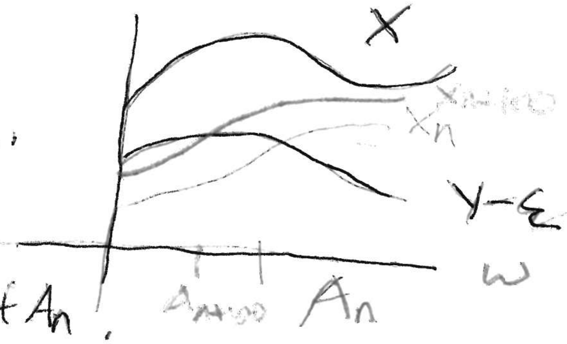
Proof] First we show that if Y is a SRV and $Y \le X$, then $E(Y) \le \lim_{n\to\infty} E[X_n]$ (*)

(Thus X_n has to get between Y and X.)

Fix $\varepsilon > 0$ and for each n, define

$A_n = \{\omega : X_n(\omega) > Y(\omega) - \varepsilon\} \in \mathcal{F}$.

Since $X_n \uparrow X > Y - \varepsilon$, $A_n \uparrow \Omega$.



Now $X_n \ge (Y-\varepsilon) I_{A_n}$ by def of A_n.

$$\therefore\ E(X_n) \underset{\substack{\uparrow \\ \text{monotonicity of SRVs}}}{\ge} E[(Y-\varepsilon) I_{A_n}] \underset{\substack{\uparrow \\ \text{linearity of SRVs}}}{=} E[Y I_{A_n}] - \varepsilon E[I_{A_n}]$$

$$\underset{\substack{\uparrow \\ I_{A_n} = 1 - I_{A_n^c}}}{=} E[Y] - E[Y I_{A_n^c}] - \varepsilon P(A_n) = C$$

$$\text{Now } E[Y I_{A_n^c}] \le \overbrace{E[I_{A_n^c}]}^{P[A_n^c]} \underbrace{\max_\omega Y(\omega)}_{\text{exists since } Y \text{ is a SRV}}.$$

monotonicity, SRV

$$Y I_{A_n^c} \le \left[\max_\omega Y(\omega)\right] I_{A_n^c}$$

$$\therefore\ E[X_n] \ge c \ge E[Y] - [\max Y]\, P(A_n^c) - \varepsilon P(A_n).$$

(Now $A_n \uparrow \Omega \Rightarrow P(A_n) \uparrow 1$, $A_n^c \downarrow \phi$, $P(A_n^c) \downarrow 0$.)

Take limits on both sides to get

$$\lim_{n\to\infty} E(X_n) \ge E[Y] - \varepsilon.$$

Since $\varepsilon > 0$ was arb,

$$\lim_{n\to\infty} E[X_n] \ge E[Y].$$

Hence $(*)$ is true.

Now for each k, $(*)$ holds with $Y = X_k'$.

Thus $E[X_k'] \le \lim_n E[X_n]$.

$$\therefore \lim_{k\to\infty} E[X_k'] \le \lim_n E[X_n].$$

$\{E[X_k']\}$ is monotone so we can take the limit

By a similar argument (symmetry)

$$\lim_{k\to\infty} E(X_k) \le \lim_{n\to\infty} E[X_n'].$$

$\square$

28) Th: a) For $X, Y \ge 0$ and $a, b \ge 0$,

$$E[aX+bY] = aE(X) + bE(Y).$$

b) If $0 \le X \le Y$, then $E(X) \le E[Y]$.

Proof] a) For SRVs $0 \le X_n \uparrow X$ and $0 \le Y_n \uparrow Y$,

the RVs $aX_n + bY_n \ge 0$ are SRVs and

$$aX_n + bY_n \uparrow \underbrace{aX+bY}_{\ge 0}.$$

$$\therefore\ E[aX+bY] \underset{\substack{\uparrow \\ E[W] \text{ for } W\ge 0}}{=} \lim_{n\to\infty} E[aX_n + bY_n] \underset{\substack{\uparrow \\ E[Z] \text{ for SRV}}}{=}$$

$$\lim_{n\to\infty}\left(aE[X_n] + bE[Y_n]\right) = a\lim_{n\to\infty} E[X_n] + b\lim_{n\to\infty} E[Y_n]$$

lim(an+bn) = lim an + lim bn if RHS exists

$$\underset{\substack{\uparrow \\ E[W] \text{ for } W\ge 0}}{=} aE[X] + bE[Y]$$

b) Since $E[W] \ge 0$ when $W \ge 0$,

$E[\underbrace{Y-X}_{W}] \ge 0$.

$$E[Y] = E[Y-X+X] \overset{\substack{X\ge 0 \\ (Y-X)\ge 0 \\ (a)}}{=} E[Y-X] + E[X] = \infty \text{ if } E(X)=\infty$$

else $E[X]<\infty$ $\quad E[Y] - E[X] = E[Y-X] \ge 0$

$\square$ LHS exists since $0 \le E[X] < \infty$.

29] Fatou's Lemma] For RVs $X_n \ge 0$,

$E[\liminf X_n] \le \liminf E[X_n]$.

Proof] For each $m \ge 1$, let $Z_m = \inf_{k\ge m} X_k \ge 0$.

Then $Z_m \uparrow \liminf_n X_n$.

Given a SRV $Y \le \underline{\lim}\, X_n$ and $\varepsilon > 0$,

$$\forall m \quad X_m \ge Z_m \ge \underbrace{(Y-\varepsilon)\, I[Z_m \ge Y-\varepsilon]}_{\begin{cases} 0 & \text{if } Z_m < Y-\varepsilon \\ Y-\varepsilon & \text{if } Z_m \ge Y-\varepsilon \end{cases}}$$

$$\therefore\ E(X_m) \ge E[\underbrace{\underbrace{(Y-\varepsilon)}_{SRV}\, \underbrace{I(Z_m \ge Y-\varepsilon)}_{SRV}}_{SRV}]$$

Can't take limits, but

$$\liminf_m E(X_m) \geq \liminf_m E[(Y-\varepsilon) I(Z_m \geq Y-\varepsilon)].$$

$W_m = (Y-\varepsilon) I(Z_m \geq Y-\varepsilon)$ is a SRV and

$W_m \uparrow Y-\varepsilon$ since $Z_m \uparrow \liminf X_n \geq Y-\varepsilon$.

Thus $I[Z_m \geq Y-\varepsilon] \uparrow I(\Omega) = 1$.

$$\therefore \underline{\lim}\, E[W_m] = \lim_m E[W_m] = E[Y-\varepsilon] = E[Y]-\varepsilon$$

and $\liminf_n E(X_n) \geq E(Y) - \varepsilon$.

Since $\varepsilon > 0$ was arb,

$\liminf E(X_n) \geq E(Y)$ for any

SRV $Y \leq \underline{\lim}\, X_n$.

In particular, this result is true for SRVs

$0 \leq \underbrace{Y_K \uparrow \liminf_n X_n}$. Thus

$$E\left[\liminf_n X_n\right] \overset{\downarrow}{=} E\left[\lim_K Y_K\right] \underset{\uparrow}{=} \lim_K E[Y_K]$$

Def $E(W)$, $\underbrace{W \geq 0}$

SRVs $Y_K \uparrow \liminf X_n \geq 0$

$$\underset{\nwarrow}{\leq} \liminf E[X_n]$$

$\lim_K E[Y_K] = \liminf E(Y_K) = \liminf_n E(Y_n) \leq \liminf E(X_n)$ $\because$

$E[Y_n] \leq \liminf E(X_n)\ \forall n$

$\square$

30] Monotone Convergence Theorem (MCT):

Let $X, X_1, X_2 \ldots$ be nonnegative RVs with $X_n \uparrow X$ ($X_n(\omega) \uparrow X(\omega)\ \forall \omega$). Then $E[X_n] \uparrow E[X]$.

(monotone continuity for nonnegative RVs)

Proof] $X_n \uparrow X$ $\therefore$ $\underbrace{E(X_n) \leq E(X)}_{\text{monotonicity w}\geq 0}$ $\forall n$

$\therefore$ $\limsup_n E(X_n) \leq E(X)$.

By Fatou's lemma, $E(X) = E[\lim X_n] = E[\liminf X_n]$
$\overset{\downarrow}{\leq} \liminf E(X_n) \leq \limsup_n E(X_n) \leq E(X)$.

Thus $\lim E[X_n] = E[X]$. $X_n \uparrow$ means $E[X_n] \overset{h\geq 0}{\leq} E[X_{n+h}]$, so $E[X_n] \uparrow E[X]$.

$\square$

31] Th $X_n \geq 0 \Rightarrow E\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} E[X_n]$.

Proof] $0 \leq Y_m = \sum_{n=1}^{m} X_n \uparrow Y = \sum_{n=1}^{\infty} X_n$.

$\therefore$ $\lim_{m\to\infty} E(Y_m) = \lim_{m\to\infty} \sum_{n=1}^{m} E(X_n) = \sum_{n=1}^{\infty} E(X_n)$

$= E(Y) = E\left(\sum_{n=1}^{\infty} X_n\right)$ by MCT.

$\square$

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32] $X = X^+ - X^-$, $|X| = X^+ + X^-$

$X^+ = \max(X, 0)$ $\quad X^- = -\min(X, 0) = \max(-X, 0)$.

33] Def RV X is integrable if $E|X| < \infty$.

34) Def! If X is integrable, then

$$E[X] = E[X^+] - E[X^-].$$

Note! If $X \geq 0$, then $E[X^-] = 0$.

35] Given $(\Omega, \mathcal{F}, P)$, the "set" (collection) of all integrable RVs is denoted by $L^1(\Omega, \mathcal{F}, P)$ or L^1. (el one)

36] Th Let X, Y be integrable ($\in L^1$).

a) Linearity; If $a, b \in \mathbb{R}$, then $aX+bY$ is integrable with $E[aX+bY] = aE(X) + bE(Y)$.

b) monotonicity; $X \leq Y \Rightarrow E(X) \leq E(Y)$.

c) $|E(X)| \leq E|X|$.

proof a) $0 \leq |aX+bY| \leq |a||X| + |b||Y|$.

By monotonicity of nonnegative RVs,

$E|aX+bY| \leq |a|E|X| + |b|E|Y| < \infty$

since $X, Y \in L^1$.

First, show $Z_1 \geq 0, Z_2 \geq 0$ and $Z_1, Z_2 \in L^1 \Rightarrow$

$E(Z_1 - Z_2) = E(Z_1) - E(Z_2)$.

Now $Z_1 - Z_2 = (Z_1 - Z_2)^+ - (Z_1 - Z_2)^-$.

Thus $Z_1 + (Z_1 - Z_2)^- = (Z_1 - Z_2)^+ + Z_2 \geq 0$,
and each term is in L^1 ($Z_1 \geq (Z_1 - Z_2)^+$, $Z_2 \geq (Z_1 - Z_2)^-$).
By linearity for a finite sum of nonnegative RVs,

$E(Z_1) + E[(Z_1 - Z_2)^-] = E[(Z_1 - Z_2)^+] + E(Z_2)$.

Thus $Z_1 - Z_2 \in L^1$ and

$E(Z_1) - E(Z_2) = E[(Z_1 - Z_2)^+] - E[(Z_1 - Z_2)^-] =$
$E(Z_1 - Z_2)$.

Second, show $E[aX] = aE(X)$.

If $a > 0$, then $(aX)^+ = \max(aX, 0) = a\max(X, 0) = aX^+$
$(aX)^- = -\min(aX, 0) = a[-\min(X, 0)] = aX^-$.

So $aX = \underbrace{(aX)^+ - (aX)^-}_{Z_1 - Z_2} = a(X^+ - X^-)$.
$(a \geq 0, W \geq 0 \Rightarrow E[aW] = aE(W))$

Similarly, $E[aX] = aE(X)$ for $a \leq 0$.
$((aX)^+ = -aX^-$ and $(aX)^- = -aX^+)$

39.5

Now $ax+by = (ax+by)^+ - (ax+by)^-$

$= (ax)^+ - (ax)^- + (by)^+ - (by)^-.$

$\therefore (ax+by)^+ + (ax)^- + (by)^- = (ax+by)^- + (ax)^+ + (by)^+.$

By linearity of finite sums of nonnegative RVs,

$E[(ax+by)^+] + E[(ax)^-] + E[(by)^-] =$

$E[(ax+by)^-] + E[(ax)^+] + E[(by)^+]$

or $E[(ax+by)^+] - E[(ax+by)^-]$

$= E[(ax)^+] - E[(ax)^-] + E[(by)^+] - E[(by)^-].$

Or $E(ax+by) = E[ax] + E(by) = aE(X) + bE(Y)$

b) $\underset{\substack{\uparrow \\ Y \geq X}}{0 \leq} E(Y-X) \overset{a)}{=} E(Y) - E(X) \Rightarrow E(Y) \geq E(X).$

c) $-|X| \leq X \leq |X| \overset{b)}{\Rightarrow} E(X) \leq E(|X|)$

and $-E(X) \leq E(|X|).$

so $|E(X)| \leq E[|X|].$ $\square$

37] Lebesgue's Dominated Convergence Theorem (LDCT): Let $X, X_1, X_2 \ldots$ be integrable with $X_n \to X$. Suppose there is integrable Y $\ni |X_n| \leq Y$ for each n. Then $\lim_{n\to\infty} E(X_n) = E[X]$.

Proof] Using nonnegativity of $Y - X_n$ and $Y + X_n$,

$$E(Y) - E(X) \underset{\text{linearity}}{=} E[Y - X] \underset{\lim(Y-X_n) = \liminf(Y-X_n) = Y-X}{=} E[\liminf (Y - X_n)] \underset{\text{Fatou}}{\leq} \liminf E(Y - X_n)$$

$$\underset{\text{linearity}}{=} E(Y) - \limsup E(X_n).$$

$E(Y)$ finite by integrability $\Rightarrow -E[X] \leq -\limsup E(X_n)$.

$$E[Y] + E(X) \underset{\text{linearity}}{=} E[Y + X] = E[\liminf(Y + X_n)] \underset{\text{Fatou}}{\leq} \liminf E(Y + X_n)$$

$$\underset{\text{linearity}}{=} E(Y) + \liminf E[X_n].$$

So $E(X) \leq \liminf E[X_n] \leq \limsup E(X_n) \leq E(X)$.

$\therefore E(X) = \lim_{n\to\infty} E[X_n]$.

$\square$

40.5

38] Def: If X is integrable and A an event, then the expectation of X over A is $E[X I_A]$.

39) $0 \le X_n \uparrow X$ if $X \ge 0$.

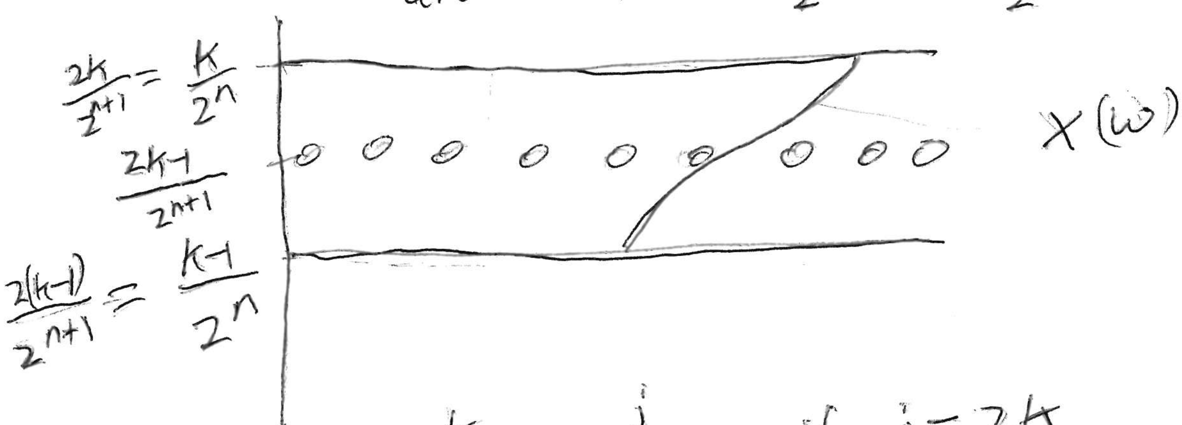
For each $n \ge 1$, let

$$X_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} I\left[\frac{k-1}{2^n} < X \le \frac{k}{2^n}\right] + n\, I(X > n).$$

Then on the event $\left\{w : \frac{k-1}{2^n} < X(w) \le \frac{k}{2^n}\right\}$, $= A_{nk}$

$X_n(w) = \frac{k-1}{2^n}$, $X_n(w) \le X(w)$ $\forall w$ and $\forall n$.

and $X_{n+1}(w) = \frac{k-1}{2^n}$ or $\frac{2k-1}{2^{n+1}} > X_n(w)$.



$$\frac{k}{2^n} = \frac{j}{2^{n+1}} \quad \text{if } j = 2k$$

$$\left(A_{nk} = \underbrace{\left\{w : \frac{k-1}{2^n} < X(w) \le \frac{2k-1}{2^{n+1}}\right\}}_{X_{n+1} = \frac{k-1}{2^n} \text{ here}} \cup \underbrace{\left\{w : \frac{2k-1}{2^{n+1}} < X(w) < \frac{k}{2^n}\right\}}_{X_{n+1} = \frac{2k-1}{2^{n+1}} \text{ here}} \right)$$

40] a) $X \geq 0$, $S(x) = 1 - F(x)$

$E(X) = \int_0^\infty S(x)dx$ which is finite

if $xS(x) \to 0$ as $x \to \infty$.

b) $E(X) = \sum_x x P(X=x)$ where the sum

is over the finite range of X,

Need more formulas for $E[h(X)]$.

41] Let $\underset{\sim}{X}$ have cdf F and let

$h(\underline{t}) : \mathbb{R}^k \to \mathbb{R}^j$ be integrable with

$1 \leq j \leq k$. Then the Lebesgue-Stieltjes

integral $E[h(\underset{\sim}{X})] = \int h(\underline{t}) dF(\underline{t})$.

The integral is a linear operator wrt

both h and F: $\int \sum_{i=1}^n h_i(\underline{t}) dF(\underline{t}) = \sum_{i=1}^n \int h_i(\underline{t}) dF(\underline{t})$

and $\int h(\underline{t})\, d\left[\sum_{i=1}^n \pi_i F_i(\underline{t})\right] = \sum_{i=1}^n \pi_i \int h(\underline{t})\, dF_i(\underline{t})$

where $0 \leq \pi_i \leq 1$ and $\sum_{i=1}^n \pi_i = 1$ so $F(\underline{t}) = \sum_{i=1}^n \pi_i F_i(\underline{t})$

is a cdf.

If X is a RV, then $E[h(X)] = \int h(t)\, dF(t)$ 41.5

and the integral exists if $h(x)$ is integrable or if $h(x) \geq 0$.

42) The dist of a $1 \times k$ random vector $\underline{X}$ is a <u>mixture distribution</u> if the cdf of X is $F_{\underline{X}}(\underline{t}) = \sum_{j=1}^{J} \pi_j F_{\underline{U}_j}(\underline{t})$ where

$0 \leq \pi_j \leq 1$, $\sum_{j=1}^{J} \pi_j = 1$, $J \geq 2$, and

$F_{\underline{U}_j}(\underline{t})$ is the cdf of a $1 \times k$ random vector $\underline{U}_j$. Then X has a mixture distribution of the $\underline{U}_j$ with probabilities π_j.

If X is a RV, then

$$F_X(t) = \sum_{j=1}^{J} \pi_j F_{U_j}(t).$$

43) <u>Expected Value Theorem</u>: Assume all expected values exist. Let $d\underline{x} = dx_1 \cdots dx_k$. Let $\mathcal{X}$ be the <u>support of X</u> $= \{\underline{x} : f(\underline{x}) > 0\}$ or $\{\underline{x} : p(\underline{x}) > 0\}$.