a) If $\underline{X}$ has (joint) pdf $f(\underline{x})$, then

$$E[h(\underline{X})] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(\underline{x}) f(\underline{x}) d\underline{x} =$$

$$\int \cdots \int_{\mathcal{X}} h(\underline{x}) f(\underline{x}) d\underline{x}.$$

$$E(\underline{X}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underline{x} f(\underline{x}) d\underline{x} = \int \cdots \int_{\mathcal{X}} \underline{x} f(\underline{x}) d\underline{x}.$$

b) If X has pdf $f(x)$, then

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x) dx = \int_{\mathcal{X}} h(x) f(x) dx$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{\mathcal{X}} x f(x) dx.$$

c) If $\underline{X}$ has (joint) pmf $p(\underline{x})$, then

$$E[h(\underline{X})] = \sum_{x_1} \cdots \sum_{x_k} h(\underline{x}) p(\underline{x}) = \sum_{\underline{x} \in \mathcal{X}} h(\underline{x}) p(\underline{x})$$

$$E(\underline{X}) = \sum_{x_1} \cdots \sum_{x_k} \underline{x} p(\underline{x}) = \sum_{\underline{x} \in \mathcal{X}} \underline{x} p(\underline{x}).$$

d) If X has pmf $p(x)$, then

$$E[h(X)] = \sum_{x} h(x) p(x) = \sum_{x \in \mathcal{X}} h(x) p(x)$$

$$E(X) = \sum_{x} x p(x) = \sum_{x \in \mathcal{X}} x p(x)$$

e) Suppose $\underline{X}$ has a mixture dist as in 42), and the $E[h(\underline{X})]$ and $E[h(\underline{U}_j)]$ exist. Then

$$E[h(\underline{X})] = \sum_{j=1}^{J} \pi_j E[h(\underline{U}_j)] \quad \text{and}$$

$$E[\underline{X}] = \sum_{j=1}^{J} \pi_j E[\underline{U}_j].$$

f) If $k=1$ in e), then

$$E[h(X)] = \sum_{j=1}^{J} \pi_j E[h(U_j)], \quad E[X] = \sum_{j=1}^{J} \pi_j E[U_j].$$

Note: a) – d) show that the calculus based expectations work for continuous and discrete random vectors. and RVs.

Proof of e) $E[h(\underline{X})] = \int h(\underline{t})\, dF(\underline{t})$

$$= \int h(\underline{t})\, d\left[\sum_{j=1}^{J} \pi_j F_{\underline{U}_j}(\underline{t})\right] \overset{\text{linearity of } \int \text{ and } dF}{=}$$

$$\sum_{j=1}^{J} \pi_j \left[\int h(\underline{t})\, dF_{\underline{U}_j}(\underline{t})\right] = \sum_{j=1}^{J} \pi_j E[h(\underline{U}_j)].$$

44) There is a 1-1 correspondence between $F(x) = F_X(x)$ and the induced Probability P_X where $P_X(B) = P(X^{-1}(B))$, $B \in \mathbb{B}(\mathbb{R})$.

Hence $\int h \, dP_X = \int h(t) \, dF(t) = E_F[h]$

where h is a random variable on $(\mathbb{R}, \mathbb{B}(\mathbb{R}), P_X)$.

45) $P_X((a,b]) = P_F((a,b]) = F(b) - F(a)$ where $F = F_X$.

46) A function h is <u>integrable wrt F</u> if $\int_{-\infty}^{\infty} |h(x)| \, dF(x) < \infty$. Then

$$\int_{-\infty}^{\infty} h(x) \, dF(x) = \int_{-\infty}^{\infty} h^+(x) \, dF(x) - \int_{-\infty}^{\infty} h^-(x) \, dF(x).$$

The integral of h over B wrt F is, for $B \in \mathbb{B}(\mathbb{R})$,

$$\int_B h(x) \, dF(x) = \int_{-\infty}^{\infty} h(x) \, I_B(x) \, dF(x).$$

47) Properties a) If $h(x) \equiv c$, then

$$E[h(X)] = E[c] = \int_{-\infty}^{\infty} c \, dF(x) = c.$$

b) For $B \in \mathbb{B}(\mathbb{R})$, $E[I_B(X)] = \int I_B(x)\, dF(x) = P_F(B)$.

c) Linearity: If $g, h \geq 0$ and $a, b \geq 0$ or if g, h are integrable with $a, b \in \mathbb{R}$, then

$$E[a\, g(X) + b\, h(X)] = \int (a\, g(x) + b\, h(x))\, dF(x)$$

$$= a\, E[g(X)] + b\, E[h(X)].$$

d) monotonicity: if $0 \leq g \leq h$ or g and h are integrable with $g \leq h$, then

$$E[g(X)] = \int g(x)\, dF(x) \leq \int h(x)\, dF(x) = E[h(X)].$$

e) Fatou's lemma: If $h_n \geq 0$ $\forall n$, then

$$E[\liminf h_n(X)] = \int [\liminf h_n(x)]\, dF(x) \leq$$

$$\liminf \int h_n(x)\, dF(x) = \liminf E[h_n(X)].$$

f) MCT: If $0 \leq h_n \uparrow h$, then

$$E[h_n(X)] = \int h_n(x)\, dF(x) \uparrow \int h(x)\, dF(x) = E[h(X)].$$

g) LDCT: If $g_n \to g$ and there is integrable

$h \ni |g_n| \le h \quad \forall n,$ then

$$E[g_n(X)] = \int g_n(x)\,dF(x) \to \int g(x)\,dF(x) = E[g(X)].$$

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48] $\int h(x)\,dF(x) = \int h(x)\,d\underbrace{\mu}_{\text{Lebesgue measure}}$

49) $\dfrac{dF(x)}{dx} = f(x)$ and $dF(x) = f(x)dx$ if pdf exists.

50] Fix $(\Omega, \mathcal{F}, P)$. $P_X = P_F$

is defined by $P_X(B) = P(X^{-1}(B))$ $\forall B \in \mathbb{B}(\mathbb{R})$.

$(\mathbb{R}, \mathbb{B}(\mathbb{R}), P_X)$ is a prob space. $E_P[h(X)] =$

$E[h(X)] = \int_\Omega h(X)\,dP = \int_{\mathbb{R}} h(x)\,dF(x) \triangleq \int_{\mathbb{R}} h\,dP_X$

$= E_F[h] = E_{P_X}[h]$. $W = h(X)$ is a RV on $(\Omega, \mathcal{F}, P)$

and $Z = h$ is a RV on $(\mathbb{R}, \mathbb{B}(\mathbb{R}), P_X)$.

Show $\int h(X)\,dP = \int h\,dP_X$.

Let h be a simple function RV on $(\mathbb{R}, \mathbb{B}(\mathbb{R}), P_X)$. Then $h(t) = \sum_{i=1}^{n} a_i I_{A_i}(t)$

and $$h(X(\omega)) = \sum_{i=1}^{n} a_i I_{A_i}(X(\omega)).$$

$= \sum_{i=1}^{n} a_i I_{X^{-1}(A_i)}(\omega)$ is a SRV wrt $(\Omega, \mathcal{F}, P)$. Then

$$E_F(h) = \int h \, dP_F = \sum_{i=1}^{n} a_i P_F(A_i) = \sum_{i=1}^{n} a_i P(X^{-1}(A_i))$$

def of $P_F = P_X$

$E_P[h(X)] = \sum_{i=1}^{n} a_i P(X^{-1}(A_i)) = \int h \, dP_F$ for any simple function h on $\mathbb{R}$.

Since $E_P[h(X)]$ and $E_F[h]$ agree for simple functions, they agree for measurable nonnegative and general functions h by the properties of expectation provided $h(X)$ is integrable.

51] Th: If h is a Borel measurable function on $\mathbb{R}$ and if X is a RV with cdf F, then the RV $h(X)$ is integrable iff

$\int |h(x)|\, dF(x) < \infty$, in which case

$$E[h(X)] = E_P[h(X)] = \int h(x)\, dF(x)$$

$$= \int h\, dP_X = E_X[h] = E_F[h].$$

If X has a pdf, then $dF(x) = f(x)dx$.

More on measures and measurability

1] Don't treat convergence of sets like convergence of functions.

$A_n \to A \quad \Rightarrow \quad \limsup A_n = \liminf A_n \Rightarrow$

if $\omega \in A_n$ for infinitely many n, then $\omega \in A_n$ for all but finitely many n.

ex] a) $A_1 \subseteq A_2 \subseteq \ldots \Rightarrow A_n \uparrow A = \bigcup_{n=1}^{\infty} A_n$

b) $A_1 \supseteq A_2 \supseteq \ldots \Rightarrow A_n \downarrow A = \bigcap_{n=1}^{\infty} A_n$.

Proof a) For each n, $A = \bigcup_{k=n}^{\infty} A_k$. $\therefore \limsup A_n = A$.

For each n, $\bigcap_{k=n}^{\infty} A_k = A_n$. $\therefore \liminf A_n = \bigcup_{n=1}^{\infty} A_n = A$.

b) see Hw 6 1.

$a<b$ 45.5

ex] a) $(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$ $= \bigcap_{n=1}^{\infty} A_n, \ A_n \downarrow I$

b) $(a,b) = \bigcup_{n=1}^{\infty} (a, b-\frac{1}{n}]$ $= \bigcup_{n=1}^{\infty} A_n, \ A_n \uparrow I$

c) $(a,b) = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b-\frac{1}{n}]$ $= \bigcup_{n=1}^{\infty} A_n, \ A_n \uparrow I$

d) $[a,b] = \bigcap_{n=1}^{\infty} [a, b+\frac{1}{n})$ $= \bigcap_{n=1}^{\infty} A_n, \ A_n \downarrow I$

e) $\{a\} = \bigcap_{n=1}^{\infty} [a, a+\frac{1}{n})$ $= \bigcap_{n=1}^{\infty} A_n, \ A_n \downarrow I$

f) $\bigcup_{n=1}^{\infty} [a, b-\frac{1}{n}] = [a,b)$ $\underbrace{= \bigcup_{n=1}^{\infty} A_n, \ A_n \uparrow I}_{A_n \to A = I}$

Note i) $a \pm \frac{1}{n} \to a$, but sets are not functions

ii) Typically want to show that open, closed and half open intervals can be written as a countable union or intersection of intervals of another type. Then $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C})$ where $\mathcal{C}$ is a class of intervals

some proofs: a) $(a,b] \subseteq A = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$ since $b \in (a, b+\frac{1}{n})$ $\underline{\forall n}$.

For any $\varepsilon > 0$, $(a, b+\varepsilon] \not\subseteq A$ since $b+\frac{1}{n} < b+\varepsilon$ for large enough n.

$n \in \mathbb{N}$ so $n = \infty$ never occurs

b) $b \notin \bigcup_{n=1}^{\infty} (a, b-\frac{1}{n}] = A$ since $b \notin (a, b-\frac{1}{n}]$ $\forall n$

$(a, b-\frac{1}{n}] = \phi$ if $b - \frac{1}{n} \leq a$.

For any $\varepsilon > 0 \ni b - \varepsilon > a$, $(a, b-\varepsilon] \in A$

since $b - \frac{1}{n} > b - \varepsilon$ for large enough n, say $n > N_\varepsilon$.

($\therefore$ $b - \varepsilon \in A$ all but finitely many times.)

c) $a, b \notin A$ since $a, b \notin [a + \frac{1}{n}, b - \frac{1}{n}]$ $\forall n \in \mathbb{N}$,

then proof is similar to b).

d) see a)

e) $a \in A$ but $a + \varepsilon \notin A$ $\forall \varepsilon > 0$

f) see b)

2] Th! Let $X: \Omega \to \mathbb{R}$. X is a measurable function iff X is a RV iff any one of the following conditions hold.

i) $X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$ $\forall B \in \mathbb{B}(\mathbb{R})$

ii) $X^{-1}((-\infty, t]) = \{\omega : X(\omega) \leq t\} \in \mathcal{F}$ $\forall t \in \mathbb{R}$

iii) $X^{-1}((-\infty, t)) = \{\omega : X(\omega) < t\} \in \mathcal{F}$ $\forall t \in \mathbb{R}$

iv) $X^{-1}([t,\infty)) = \{\omega : X(\omega) \geq t\} \in \mathcal{F} \quad \forall t \in \mathbb{R}$

v) $X^{-1}((t,\infty)) = \{\omega : X(\omega) > t\} \in \mathcal{F} \quad \forall t \in \mathbb{R}$

Proof sketch: i) $\Rightarrow$ ii) – v)

ii) and v) are complementary since $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

iii) " iv) "

Let $\mathcal{C}$ be the collection of all subsets C of $\mathbb{B}(\mathbb{R})$

$\ni X^{-1}(C) = \{\omega : X(\omega) \in C\} \in \mathcal{F}$.

It can be shown that $\mathcal{C}$ is a σ-field.

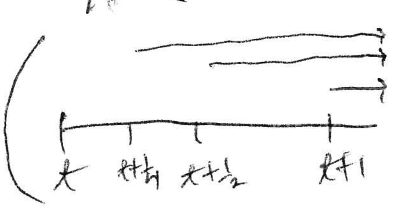
Now (t,∞), $[t,\infty)$, $(-\infty,t)$, $(-\infty,t] \in \mathcal{C} \quad \forall t \in \mathbb{R}$.

$\therefore$ $\mathbb{B}(\mathbb{R})$ = smallest σ field containing such intervals $\subseteq \mathcal{C}$. Hence $\mathcal{C} \subseteq \mathbb{B}(\mathbb{R}) \Rightarrow \mathcal{C} = \mathbb{B}(\mathbb{R})$.

v) $\Rightarrow$ iv) since $\{\omega : X(\omega) \geq t\} = \bigcap_{n=1}^{\infty} \overbrace{\{\omega : X(\omega) > t - \tfrac{1}{n}\}}^{\in \mathcal{F}}$

iv) $\Rightarrow$ v) since $\{\omega : X(\omega) > t\} = \bigcup_{n=1}^{\infty} \underbrace{\{\omega : X(\omega) \geq t + \tfrac{1}{n}\}}_{\in \mathcal{F}}$

ii) $\Leftrightarrow$ iii) is similar.

(

$\{X \geq t+1\} \supseteq \{X \geq t+\tfrac{1}{2}\} \supseteq \cdots$)

t is in none of these sets but $t+\varepsilon$, $\varepsilon > 0$ is eventually

3] Let X and Y be RVs.

a) $aX+bY$ is a RV for all $a,b \in \mathbb{R}$

b) $\max(X,Y)$ is a RV

c) $\min(X,Y)$ is a RV

d) XY is a RV

e) X/Y is a RV if $Y(\omega) \neq 0$ $\forall \omega \in \Omega$.

Proof] a) For each t

$$\{X+Y < t\} = \bigcup_{\underbrace{r \in \mathbb{Q}}_{\text{rationals}}} \left[\{X<r\} \cap \{Y < t-r\}\right] \in \mathcal{F}$$

sums of RVs are RVs

countable union

If $a>0$, for each t, $\{aX \le t\} = \{X \le \frac{t}{a}\} \in \mathcal{F}$.

If $a<0$, $\{aX \le t\} = \{X \ge \frac{t}{a}\} \in \mathcal{F}$.

If $a=0$, $aX \equiv 0$ and a constant is a RV.

b) $\underbrace{\{\max(X,Y) \le t\} = \{X \le t\} \cap \{Y \le t\}}_{\text{both are } \le t} \in \mathcal{F}$.

< at least 1 $\leq t$ >

c) $\{\min(X,Y) \leq t\} = \{X \leq t\} \cup \{Y \leq t\} \in \mathfrak{F}$.

d) 1st X^2 is a RV since for $t \geq 0$,

$$\{X^2 \leq t\} = \{-\sqrt{t} \leq X \leq t\} = \{X \leq \sqrt{t}\} - \{X < -\sqrt{t}\} \in \mathfrak{F}.$$

$\{X^2 \leq t\} = \phi \in \mathfrak{F}$ for $t < 0$.

Then $XY = \frac{1}{2}\left[(X+Y)^2 - X^2 - Y^2\right] \in \mathfrak{F}$ by a),

e) $\frac{1}{Y}$ is a RV since

$$\left\{\frac{1}{Y} \leq t\right\} = \begin{cases} \{Y \geq \frac{1}{t}\} \cup \{Y \leq 0\}, & t \geq 0 \\ \{Y \geq \frac{1}{t}\}, & t < 0. \end{cases}$$

(If $t = 0$, $\frac{1}{Y} \leq 0$ iff $Y < 0$ iff $Y \leq 0$ since $Y(\omega) \neq 0$ $\forall \omega$.)

$\therefore$ $\frac{X}{Y} = X \frac{1}{Y}$ is a RV by d).

ex] $X^+ = \max(X,0)$ and $X^- = -\min(X,0)$ are RVs if X is a RV. So $X^+ + X^- = |X|$ is a RV.

4] Let $X_1, X_2, \ldots$ be RVs on $(\Omega, \mathcal{F}, P)$.

Then i) $\sup_n X_n$, ii) $\inf_n X_n$, iii) $\limsup_n X_n$, iv) $\liminf X_n$

are RVs. v) If $\lim_n X_n = X$ $\quad (X(\omega) = \lim_{n\to\infty} X_n(\omega)\ \forall \omega)$

then X is a RV.

Proof] i) For each t, $\{\sup_n X_n \le t\} = \bigcap_{n=1}^{\infty} \{X_n \le t\} \in \mathcal{F}$.

ii) $\{\inf_n X_n \ge t\} = \bigcap_{n=1}^{\infty} \{X_n \ge t\} \in \mathcal{F}$.

iii) $\limsup_n X_n = \inf_K \sup_{m \ge K} X_m = \inf_K Y_K$ $\underset{\text{by ii)}}{\underset{\uparrow}{\text{is}}}$ a RV.

iv) $\liminf_n X_n = \sup_K \inf_{m \ge K} X_m = \sup_K W_K$ $\underset{\text{by i)}}{\underset{\uparrow}{\text{is}}}$ a RV.

v) $X = \limsup_n X_n = \liminf X_n$ is a RV.

5] If $X_1, X_2, \ldots$ are RVs and $\sum_{n=1}^{\infty} X_n \to X$,

$\left(\sum_{n=1}^{\infty} X_n(\omega) \to X(\omega)\ \ \forall \omega\right)$, then X is a RV.

Proof $Y_m = \underbrace{\sum_{n=1}^{m} X_n}_{\text{RV by induction and 3a)}}$ and $\lim_n Y_m = X$.

Fix $(\Omega, \mathcal{F}, P)$.

6] Let $X_1, \ldots, X_K$ be RVs and

$h: \mathbb{R}^k \to \mathbb{R}$ be measurable.

Then $Y = h(X_1, \ldots, X_K)$ is a RV.

7] If $h: \mathbb{R}^k \to \mathbb{R}$ is continuous, then

h is measurable.

8] problem 13.3: A monotone real function is measurable.

9] Let h be a measurable function $h: \mathbb{R} \to \mathbb{R}$. Denote the Lebesgue integral by $\int h \, \underbrace{d\mu_L}_{\text{Lebesgue measure}} = \int_{-\infty}^{\infty} h(x)\,dx$.

Then h is integrable if $\int_{-\infty}^{\infty} |h(x)|\,dx < \infty$.

10] Let $f(x) \geq 0$ be an integrable pdf of a RV with cdf F. Then

$P_X(B) = P_F(B) = \int_B f(x)\,dx$.

So many prob dists can be obtained with Lebesgue integration. $F(b) - F(a) = \int_a^b f(x)\,dx = P_F((a,b])$.