11] Typically the Lebesgue integral is the Riemann integral, and

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1$$ so the

pdf $f(x)$ is integrable.

see 14]

12] Recall that RVs $X_1, \dots, X_K$ are ind

p 267-8 iff $P[X_1 \in B_1, \dots, X_K \in B_K] = \prod_{i=1}^{K} P(X_i \in B_i)$

for any $B_1, \dots, B_K \in \mathbb{B}(\mathbb{R})$.

iff $F(x_1, \dots, x_K) = F_{X_1}(x_1) \cdots F_{X_K}(x_K)$ for all

real $x_1, \dots, x_K$.

If $X_1, \dots, X_K$ have joint pdf $f(\underline{x})$, then

$X_1, \dots, X_K$ are ind iff $f(\underline{x}) =$

$f(x_1, \dots, x_K) = f_{X_1}(x_1) \cdots f_{X_K}(x_K)$ for all real $x_1, \dots, x_K$.

§3.18 Product measure and Fubini's Th.

13] Let $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ be two prob spaces.

Let the cartesian product = cross product 49.5

$\Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) ;\ \omega_1 \in \Omega_1, \omega_2 \in \Omega_2)$,

The product of $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted by $\mathcal{F}_1 \times \mathcal{F}_2$,

is $\mathcal{F}_1 \times \mathcal{F}_2 = \sigma(A)$ where

$A = \{ \underbrace{A_1 \times A_2}_{\text{cross products}} : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2 \}$.

14) Th; There is a unique probability

$P = P_1 \times P_2$ such that $P(A_1 \times A_2) = P_1(A_1) P_2(A_2)$

for all $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$.

15) $P = P_1 \times P_2$ is the product of P_1 and P_2.

The product probability space is

$(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \times P_2)$

16) The above can be extended to

$(\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, \ldots, n$.

Denote $P_1 \times \cdots \times P_n$ by $\prod_{i=1}^{n} P_i$.

ex3 If $(\Omega_i, \mathcal{F}_i, P_i) = (\mathbb{R}, \mathbb{B}(\mathbb{R}), P_i)$

then the product space is $\left(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n), \prod_{i=1}^n P_i\right)$.

If $(\Omega_i, \mathcal{F}_i, P_i) = (\mathbb{R}, \mathbb{B}(\mathbb{R}), P_{X_i})$

then the product space is $\left(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n), \prod_{i=1}^n P_{X_i}\right)$.

14) Let X_i be defined on $(\mathbb{R}, \mathbb{B}(\mathbb{R}), P_{X_i})$

$X_1, \dots, X_n$ ind. Then

$F_{\underline{X}}(\underline{t}) = P(X_1 \le t_1, \dots, X_n \le t_n) =$

$P((-\infty, t_1] \times \dots \times (-\infty, t_n]) = \prod_{i=1}^n P_{X_i}((-\infty, t_i])$

$= \prod_{i=1}^n F_{X_i}(t_i)$ where $P = \prod_{i=1}^n P_{X_i}$.

Hence $(\Omega, \mathcal{F}, P) = \left(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n), \prod_{i=1}^n P_{X_i}\right)$

is the prob space for $\underline{X} = (X_1, \dots, X_n)$.

$P_{X_i}(B) = P(X_i \in B) = P(\mathbb{R} \times \dots \times \mathbb{R} \times B \times \mathbb{R} \times \dots \times \mathbb{R})$

$= P_{X_1}(\mathbb{R}) \cdots P_{X_{i-1}}(\mathbb{R})\, P_{X_i}(B)\, P_{X_{i+1}}(\mathbb{R}) \cdots P_{X_n}(\mathbb{R})$.

15] Let $\int f \, d\mu = \int f(x) \, d\mu(x)$.

Then $\iint_{\Omega_1 \times \Omega_2} f(x_1, x_2) \, d[P_1 \times P_2(x_1, x_2)]$

$= \int_{\Omega_1} \left[\int_{\Omega_2} f(x_1, x_2) \, dP_2(x_2) \right] dP_1(x_1)$

$= \int_{\Omega_2} \left[\int_{\Omega_1} f(x_1, x_2) \, dP_1(x_1) \right] dP_2(x_2)$.

16] Fubini's Theorem; a) Assume $f \geq 0$.

Then $\int_{\Omega_2} f(x_1, x_2) \, dP_2(x_2)$ is measurable $\mathcal{F}_1$,

$\int_{\Omega_1} f(x_1, x_2) \, dP_1(x_1)$ is measurable $\mathcal{F}_2$,

and 15] holds.

b) Assume f is integrable wrt $P_1 \times P_2$.

Then $\int_{\Omega_2} f(x_1, x_2) \, dP_2(x_2)$ is finite a e and measurable $\mathcal{F}_1$ ae,

$\int_{\Omega_1} f(x_1, x_2) \, dP_1$ is finite ae and measurable $\mathcal{F}_2$ ae, and 15] holds.

17] 16 a) = Tonelli's Th

p238 18] If f is not nonnegative, compute one of the integrals in 16] with $|f|$ replacing f. If the result is finite, compute one of the iterated ~~double~~ integrals in 15] with f. (If the result is infinite, then f is not integrable wrt $P_1 \times P_2$.)

19] The exchange of integrals 15] holds if $f \geq 0$ or if $\iint |f(x,y)|\, d[P_1 \times P_2] < \infty$.

20] Fubini's th allows the computation of $E[h(X_1, X_2)]$ using integrals in 15] if X_1 and X_2 are independent.

21] A <u>Product measure</u> μ satisfies

$$\mu\left(\prod_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \mu_i(A_i).$$

exs] i) $\mu = \prod_{i=1}^{n} P_{X_i}$ when the X_i's are ind.

ii) Let λ be the Lebesgue measure on $\mathbb{R}^2$

and μ_L $\mathbb{R}$.

Then $\lambda(A \times B) = \mu_L(A)\, \mu_L(B)$, $A, B \in \mathbb{B}(\mathbb{R})$.

iii) Let ν be the counting measure on $\mathbb{Z}^2$

μ_c $\mathbb{Z}$.

$\mu_c(A) = \#$ of points in A.

$\nu(A \times B) = 12$

Then $\nu(A \times B) = \mu_c(A)\, \mu_c(B)$, $A, B \subseteq \mathbb{Z}$.

(counting measure μ_c is σ-finite if Ω is countable. p158)

22] Fubini's theorem holds for product measures $\mu = \prod_{i=1} \mu_i$ if the μ_i are σ-finite.

23] Fubini's theorem for Lebesgue integrals:

Let $C = \{(x, y) : a \le x \le b,\ c \le y \le d\} = [a,b] \times [c,d]$.

Let $g(x, y)$ be measurable and Lebesgue integrable. Then

$$\iint_C g(x,y)\, dx\,dy = \int_c^d \left[\int_a^b g(x,y)\, dx \right] dy$$

$$= \int_a^b \left[\int_c^d g(x,y)\, dy \right] dx.$$

24] The result in 23] can be extended to where the limits of integration are infinite and to $n > 2$ integrals. Using $g(x,y) = h(x,y)\,f(x,y)$ where f is a pdf gives $E[h(X,Y)]$.

Independent RVs

p267 25] RVs $X_1, \dots, X_n$ are <u>independent</u> if

$$P(X_1 \in B_1, \dots, X_n \in B_n) = \prod_{i=1}^{n} P(X_i \in B_i)$$

for any $B_i \in \mathbb{B}(\mathbb{R})$. An infinite set of RVs is <u>independent</u> if each finite subset is ind.

26] $X_1, \dots, X_n$ are ind

a) iff $F_{X_1,\dots,X_n}(t_1,\dots,t_n) = \prod_{i=1}^{n} F_{X_i}(t_i) \quad \forall\, t_1,\dots,t_n \in \mathbb{R}$.

b) If pdfs exist, iff

$$f_{X_1,\dots,X_n}(t_1,\dots,t_n) = f_{X_1}(t_1) \cdots f_{X_n}(t_n) \quad \forall\, t_1,\dots,t_n \in \mathbb{R}.$$

c) If pmfs exist, iff

$$P_{X_1,\dots,X_n}(t_1,\dots,t_n) = P_{X_1}(t_1) \cdots P_{X_n}(t_n) \quad \forall\, t_1,\dots,t_n \in \mathbb{R}$$

Note: if the discrete RVs $X_1, \dots, X_n$ take on values in a countable set $\mathcal{C}$, then $X_1, \dots, X_n$ are ind iff

$$p_{X_1,\dots,X_n}(t_1,\dots,t_n) = \prod_{i=1}^{n} p_{X_i}(t_i) \quad \forall\, t_1,\dots,t_n \in \mathcal{C}.$$

p267 27] If $X_1, \dots, X_n$ are ind and $\underset{\sim}{X} = (X_1, \dots, X_n)$ has dist $P_{\underset{\sim}{X}}(C) = P(\underbrace{\underset{\sim}{X}^{-1}(C)}_{\{\omega : \underset{\sim}{X}(\omega) \in C\}})$ for $C \in \mathbb{B}(\mathbb{R}^n)$, then $P_{\underset{\sim}{X}}(\underbrace{B_1 \times \cdots \times B_n}_{\text{cross product}})$

$$= P(\underset{\sim}{X} \in B_1 \times \cdots \times B_n) = P(X_1 \in B_1, \dots, X_n \in B_n)$$

$$= \prod_{i=1}^{n} P(X_i \in B_i) = \prod_{i=1}^{n} P_{X_i}(B_i).$$

Thus $P_{\underset{\sim}{X}}$ is a product probability.

p284 28] $E[g_1(X_1) \cdots g_n(X_n)] = E[g_1(X_1)] \cdots E[g_n(X_n)]$

if g_i is a function of X_i alone, $X_1, \dots, X_n$ are ind, and the expected values exist.

29) Suppose $X \perp\!\!\!\perp Y$. Then for $t \in \mathbb{R}$,

$$\underset{\text{cdf}}{F_{X+Y}}(t) = \int F_X(t-y)\,dF_Y(y) = \int F_Y(t-x)\,dF_X(x)$$

$$= E_Y\left[F_X(t-Y)\right] = E_X\left[F_Y(t-X)\right].$$

If X has pdf $f_X(x)$ and Y has pdf $g_Y(y)$,

then
$$\overset{\text{pdf}}{f_{X+Y}}(t) = \int_{-\infty}^{\infty} f_X(t-y)\,f_Y(y)\,dy = E_Y\left[f_X(t-Y)\right]$$

$$= \underbrace{\int_{-\infty}^{\infty} g_Y(t-x)\,f_X(x)\,dx}_{\text{convolution formulas}} = E_X\left[g_Y(t-X)\right]$$

Large Sample Theory

Approximating the dist of $\bar{X}_n$, CLT, WLLN

SLLN, characteristic function $\xrightarrow[L]{D}$ $\xrightarrow{P}$ $\xrightarrow[\text{mean}]{L_1}$

$\xrightarrow[\substack{\text{wp1} \\ \text{ae} \\ \text{as}}]{}$

p367 1] know **Central Limit Theorem** (CLT):

Let $X_1, \dots, X_n$ be iid with mean $E(X_i) = \mu$ and variance $V(X_i) = \sigma^2$.

Let the sample mean $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

Then $\sqrt{n}\,(\overline{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$.

Note: Hence $\sqrt{n}\left(\frac{\overline{X}_n - \mu}{\sigma}\right) = \underbrace{\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}}_{\text{zscore of } \overline{X}_n} = \sqrt{n}\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{n\sigma}\right)$

$$= \underbrace{\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{n\sigma}\right)}_{\text{z score of } \sum_{i=1}^{n} X_i} \xrightarrow{D} N(0,1).$$

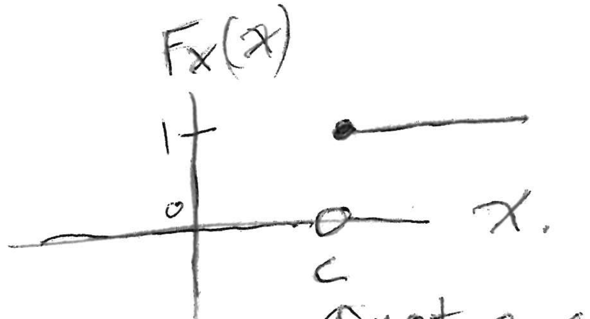
Note: The CLT will be proved later.

p14, 25
p192
335
2] Def If F_n and F are cdfs, F_n **converges weakly** to F, written $F_n \xrightarrow{w} F$, if $\lim_n F_n(x) = F(x)$ at every continuity point of F.

p338 3] know Def Let X_n and X be RVs with cdfs F_n and F. X_n converges in distribution to X or X_n converges in law to X, written $X_n \xrightarrow{D} X$ or $X_n \xrightarrow{L} X$, if F_n converges weakly to F:

$$\lim_{n \to \infty} F_n(x) = F(x)$$ at every continuity point of F.

4] X has a point mass at c if $P(X=c)=1$. Then $F_X(x) = \begin{cases} 1 & x \geq c \\ 0 & x < c \end{cases}$



often $c = \mu$ or θ.

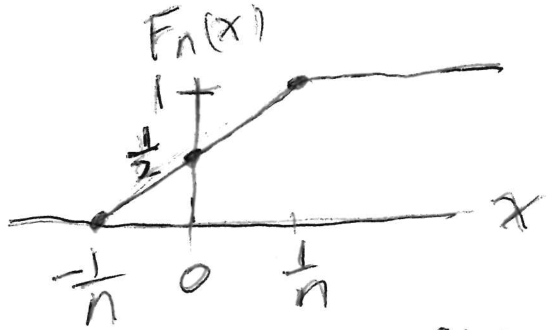
not a continuity point of F_X

p339 5] $X_n \xrightarrow{D} X$ iff $\lim_{n \to \infty} P[X_n \leq x] = P[X \leq x]$

for every x such that $\underbrace{P[X = x] = 0}_{\text{no jump in the cdf at } x}$.

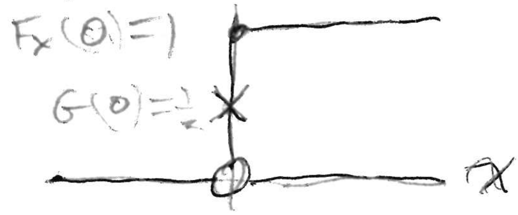
ex] Suppose $X_n \sim U\left(-\frac{1}{n}, \frac{1}{n}\right)$. Then

$$F_n(x) = \begin{cases} 0 & x \le -\frac{1}{n} \\ \frac{nx}{2} + \frac{1}{2}, & -\frac{1}{n} \le x \le \frac{1}{n} \\ 1, & x \ge \frac{1}{n} \end{cases} \qquad F_n(0) = \tfrac{1}{2} \ \forall n$$



$$F_n(x) \to \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} = G(x)$$

$F_X(x)$ and $G(x)$



$$\text{so } F_n(x) \to F_X(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$$

pointmass at 0

for each continuity point of X.

Hence $X_n \xrightarrow{D} X$. The discontinuity point of $F_X(x)$ is at $x = 0$, and $\lim_{n\to\infty} F_n(0)$ is unimportant for convergence in distribution (the limit need not exist and need not be equal to $F_X(0)$).

discontinuity point of $F_X(x)$

ex] $X_n \sim U(0,n)$. Then $F_n(t) = \begin{cases} 0 & t \le 0 \\ \frac{t}{n} & 0 \le t \le n \\ 1 & t \ge n \end{cases}$

$\lim_{n\to\infty} F_n(t) = 0 \equiv H(t) \ \forall t \in \mathbb{R}$.

X_n does not converge in dist to a RV X

since $H(t) \equiv 0$ is not a cdf and can't be changed to a cdf by modifying a countable number of discontinuity points.

6] If $X_n \xrightarrow{D} X$, then X is the limiting distribution or asymptotic distribution of X_n. Note that the distribution of X does not depend on n. ← sample size

7] For the CLT, $X \sim N(0, \sigma^2)$.

ex] If $X_1, \ldots, X_n$ are iid $U(0,1)$, find the limiting distribution of $\sqrt{n}\,(\overline{X}_n - c)$ for appropriate c. (In 580 quals. (stat) the question often asks for the limiting distribution of $\overline{X}_n$.)

Soln] $E(X_i) = \frac{0+1}{2} = \frac{1}{2}$, $V(X_i) = \frac{(1-0)^2}{12} = \frac{1}{12}$

so $\underbrace{\sqrt{n}\left(\bar{X}_n - \frac{1}{2}\right)}_{\uparrow\, c} \xrightarrow{D} \underbrace{N\left(0, \frac{1}{12}\right)}_{X}$ by CLT.

8) Several distributions are such that

$X_n = \sum_{i=1}^{n} Y_i$ where Y_i are iid.

a) $X_n \sim bin(nk, p)$, $X_n = \sum_{i=1}^{n} Y_i$, $Y_i \overset{iid}{\sim} bin(k, p)$

b) $X_n \sim poisson(n\lambda)$, $Y_i \overset{iid}{\sim} poisson(\lambda)$

c) $X_n \sim \chi^2_{np}$, $Y_i \overset{iid}{\sim} \chi^2_p$, $E(Y_i) = p$, $V(Y_i) = 2p$

d) $X_n \sim G(n\alpha, \beta)$, $Y_i \overset{iid}{\sim} G(\alpha, \beta)$, $E(Y_i) = \frac{\alpha}{\beta}$, $V(Y_i) = \frac{\alpha}{\beta^2}$
gamma$(n\alpha, \beta)$

e) $X_n \sim NB(nr, p)$, $Y_i \overset{iid}{\sim} NB(r, p)$, $E(Y_i) = \frac{r(1-p)}{p}$, $V(Y_i) = \frac{r(1-p)}{p^2}$
negative binomial

f) $X_n \sim N(n\mu, n\sigma^2)$, $Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$

g) $X_n \sim IG(n\theta, n^2\lambda)$, $Y_i \overset{iid}{\sim} IG(\theta, \lambda)$, $E(Y_i) = \theta$, $V(Y_i) = \frac{\theta^3}{\lambda}$
inverse Gaussian

Show with mgfs or characteristic functions.