Then $\sqrt{n}\left(\frac{X_n}{n} - E(Y_i)\right) \xrightarrow{D} N(0, V(Y_i))$ by the CLT,

since $\frac{X_n}{n} = \overline{Y}_n$.

ex] Let $X \sim \text{bin}(n, p)$ where $0 < p < 1$.

Find the limiting dist of $\sqrt{n}\left(\frac{X}{n} - c\right)$ for appropriate constant c.

soln] $X = X_n = \sum_{i=1}^{n} Y_i$, $Y_i \stackrel{iid}{\sim} \text{bin}(1, p)$

$E(Y_i) = p$, $V(Y_i) = p(1-p)$

$\therefore \sqrt{n}\left(\frac{X}{n} - \underset{\substack{\uparrow \\ c}}{p}\right) = \sqrt{n}(\overline{Y} - p) \xrightarrow{D} N(0, p(1-p))$.

9] Write $Y \sim X$ or $Y \stackrel{D}{=} X$ if Y and X have the same distribution.

So in the above ex, could write $X \stackrel{D}{=} \sum_{i=1}^{n} Y_i$.

<u>end exam 2 material</u>

<u>begin exam 3 material</u>

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10] Def The <u>characteristic function</u> $\varphi_X(t) =$

$E[e^{itX}] = E[\cos(tX)] + i\,E[\sin(tX)]$

where the complex number $i = \sqrt{-1}$.
$i^2 = -1$

11] If X is a random variable, then $\varphi_X(t)$ always exists and $\varphi_X(t)$, like $F_X(t)$, completely determines the distribution of X.
If the mgf exists, the mgf also completely determines the distribution of X.

12] If the mgf exists, $m_X(it) \text{"="} E(e^{itX}) = \varphi_X(t)$
and $\varphi_X(-it) \text{"="} E[e^{i(-it)X}] = E[e^{tX}] = m_X(t)$.
Technically $\varphi_X(t)$ and $m_X(t)$ are defined for $t \in \mathbb{R}$, but the above formulas work.

Know named theorems

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13] Continuity Theorem: Let X_n be a sequence of RVs with cfs $\varphi_n(t)$. Let X be a RV with cf $\varphi(t)$.

a) $X_n \xrightarrow{D} X$ iff $\varphi_n(t) \to \varphi(t)$ $\forall\, t \in \mathbb{R}$.

b) Also assume the X_n have mgfs $m_n(t)$ and X has mgf $m(t)$. Assume all mgfs m_n and m are defined on $|t| < d$ for some $d > 0$. If $m_n(t) \to m(t)$ as $n \to \infty$ for all $|t| < c$ where $0 < c < d$, then $X_n \xrightarrow{D} X$.

14) P343 Let $g: \mathbb{R} \to \mathbb{R}$.

a) Continuous mapping Theorem: If $X_n \xrightarrow{D} X$ and g is a continuous function, then $g(X_n) \xrightarrow{D} g(X)$.

b) Generalized continuous Mapping Theorem: If $X_n \xrightarrow{D} X$ and the measurable function g is such that $P[X \in C(g)] = 1$ where $C(g)$ is the set of points where g is continuous, then $g(X_n) \xrightarrow{D} g(X)$.

Note: $P[X \in D(g)] = 0$ where $D(g)$ is the set of points where g is not continuous.

ex] If $X_n \xrightarrow{D} X$, then $\frac{1}{X_n} \xrightarrow{D} \frac{1}{X}$ if X is a continuous RV since $P(X=0) = 0$ and $x = 0$ is the only discontinuity point of $g(x) = \frac{1}{x}$.

ex] Let $Y \sim N(\mu, \sigma^2)$. Then

the characteristic function of Y is

$$\varphi_Y(t) = \exp\left(it\mu - t^2\frac{\sigma^2}{2}\right).$$

The $N(\mu, 0)$ distribution has $\varphi_X(t) = \exp(it\mu)$, but does not have a pdf.

a) Find the characteristic function of X if $P(X=\mu) = 1$. So X is the point mass at μ.

Soln $E\left[e^{itX}\right] = \sum_x e^{itx} P(X=x) = e^{it\mu} \underbrace{P(X=\mu)}_{1} = e^{it\mu}$.

Hence $X \sim N(\mu, 0)$.

b) Let $X_n \sim N(\mu, \frac{1}{n}) \sim N\left(\mu, \left(\frac{1}{\sqrt{n}}\right)^2\right)$.

Then $F_n(t) = \Phi\left(\frac{t-\mu}{\sigma_n}\right) = \Phi\left[\sqrt{n}\,(t-\mu)\right]$

$$\to \begin{cases} \Phi(-\infty) = 0 & t < \mu \\ \Phi(0) = \frac{1}{2} & t = \mu \\ \Phi(\infty) = 1 & t > \mu \end{cases}$$

So $F_n(t) \xrightarrow{w} F_X(t) = \begin{cases} 0 & t < \mu \\ 1 & t \geq \mu \end{cases}$ and $X_n \xrightarrow{D} X$.

Also $\varphi_n(t) = \exp\left(it\mu - t^2\frac{1}{2n}\right) \to \exp(it\mu) = \varphi_X(t)\ \forall t$. $\therefore X_n \xrightarrow{D} X$ by the continuity th.

15] Let $X_1, \ldots, X_n$ be independent RVS with characteristic functions $\varphi_{X_j}(t)$.

Then $\varphi_{\sum_{j=1}^{n} X_j}(t) = \prod_{j=1}^{n} \varphi_{X_j}(t)$.

If the RVS have mgfs $m_{X_j}(t)$, then

$$m_{\sum_{j=1}^{n} X_j}(t) = \prod_{j=1}^{n} m_{X_j}(t).$$

proof] $\varphi_{\sum_{j=1}^{n} X_j}(t) = E\left[e^{it\sum_{j=1}^{n} X_j}\right] =$

$$E\left[e^{itX_1 + itX_2 + \ldots + itX_n}\right] \overset{\text{ind}}{=}$$

(Still works for complex RVS)

$$E\left[\prod_{j=1}^{n} e^{itX_j}\right] = \prod_{j=1}^{n} E(e^{itX_j}) = \prod_{j=1}^{n} \varphi_{X_j}(t).$$

The proof for mgfs is the same except omit the i's. $\square$ and change φ to m.

16] A sequence of RVS X_n converges in distribution to a constant c, $X_n \xrightarrow{D} c$, if $X_n \xrightarrow{D} X$ where $P(X=c)=1$. Hence X is the point mass at c.

P.22 17] Know A sequence X_n converges in probability to X, written $X_n \xrightarrow{P} X$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| < \varepsilon) = 1 \iff \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$$

$(<)$

18] Know] A sequence X_n converges in probability to a constant c, $X_n \xrightarrow{P} c$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - c| < \varepsilon) = 1 \iff \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0.$$

Note] a) often $c = \theta$ and convergence in probability is related to consistency of an estimator.

b) $X_n \xrightarrow{P} X$ iff $X_n - X \xrightarrow{P} 0$.

19] For a real number $r > 0$, X_n converges in rth mean to X, $X_n \xrightarrow{r} X$, if $E[|X_n - X|^r] \to 0$ as $n \to \infty$.

In particular, if $r = 2$, X_n converges in quadratic mean to X, $X_n \xrightarrow{2} X$, $X_n \xrightarrow{L^2} X$, $X_n \xrightarrow{qm} X$, if $E[(X_n - X)^2] \to 0$ as $n \to \infty$.

replace X by c for $X_n \xrightarrow{r} c$.

20] A sequence of RVS X_n **converges almost everywhere** to X

or **converges almost surely** to X

or **converges with probability 1** to X

if $P\left(\lim_{n\to\infty} X_n = X\right) = 1$. Denote this

convergence by $X_n \xrightarrow{ae} X$, $X_n \xrightarrow{as} X$, $X_n \xrightarrow{wp1} X$.
$(P(\{\omega : X_n(\omega) \to X(\omega)\}) = 1)$

21] X_n converges $\begin{matrix} ae \\ as \\ wp1 \end{matrix}$ to a constant c,

written $X_n \xrightarrow{ae} c$, $X_n \xrightarrow{as} c$, $X_n \xrightarrow{wp1} c$, if

$P\left(\lim_{n\to\infty} X_n = c\right) = 1$.

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22] For $X_n \xrightarrow{ae} X$, need $\{X_n\}$ and X defined on the same probability space $(\Omega, \mathcal{F}, P)$.
For $X_n \xrightarrow{D} X$, this condition is not needed:
just need $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ at all continuity points of F_X. In 18) & 21] the X_n can have different prob spaces.
$(\xrightarrow{a} c)$

Know p290

23] **Strong Law of Large Numbers (SLLN):**

If $X_1, X_2, \ldots$ are iid with $E[X_i] = \mu$ finite,

then $\overline{X}_n \xrightarrow{wp1} \mu$.

p81 ex6.3

24] Know] **Weak Law of Large Numbers (WLLN):**

If $X_1, X_2, \ldots$ are iid. with $E(X_i) = \mu$ finite,

then $\overline{X}_n \xrightarrow{P} \mu$.

memorize

Proof for when X_i also have $V(X_i) = \sigma^2$:

By Chebyshev's inequality, for every $\varepsilon > 0$,

$$P\left[|\overline{X}_n - \mu| \geq \varepsilon\right] \leq \frac{V(\overline{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty.$$

$\square$

Warning: Students often forget that $V(X_i) = \sigma^2$ is not needed: $\overline{X}_n \xrightarrow{P} \mu$ if $E(X_i) = \mu$ is finite.

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25] If $F_n \xrightarrow{w} F$ and $F_n \xrightarrow{w} G$, then $F = G$.

Hence if $X_n \xrightarrow{D} X$ and $X_n \xrightarrow{D} Y$, then $X \stackrel{D}{=} Y$.

Proof] F and G agree at their common points of continuity, hence at all but see quizzes 4–7, HW 4–7 Exam 2 review, 1

I] If X has a pmf, then $E[h(X)] = \sum_x h(x)\,p(x)$

pdf $\int_{-\infty}^{\infty} h(x)\,f(x)\,dx.$

$E[h(X)] = \int h(x)\,dF(x).$

If X has a mixture distribution of the U_i with probs π_i, then $E[h(X)] = \sum_{i=1}^{J} \pi_i E[h(U_i)]$.

a) If each U_i has pmf p_{U_i}, then X has pmf $\sum_{i=1}^{J} \pi_i\, p_{U_i}(x)$ and

$$E[h(X)] = \sum_x h(x) \sum_{i=1}^{J} \pi_i\, p_{U_i}(x)$$

$$= \sum_{i=1}^{J} \pi_i \sum_x h(x)\, p_{U_i}(x) = \sum_{i=1}^{J} \pi_i\, E[h(U_i)].$$

← badly done on HW 6

b) If each U_i has pdf f_{U_i}, then

X has pdf $f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \sum_{i=1}^{J} \pi_i F_{U_i}(x)$

$= \sum_{i=1}^{J} \pi_i f_{U_i}(x)$. $\therefore E[h(X)] =$

$$\int_{-\infty}^{\infty} h(x) \sum_{j=1}^{J} \pi_i f_{U_i}(x)\,dx = \sum_{j=1}^{J} \pi_i \int_{-\infty}^{\infty} h(x) f_{U_i}(x)\,dx$$

$$= \sum_{i=1}^{J} \pi_i\, E[h(U_i)].$$

II) CLT: Y_i iid, $E(Y)=E(Y_i)=\mu$, $V(Y)=V(Y_i)=\sigma^2$

$$\sqrt{n}\left[\bar{Y}_n - E(Y)\right] \xrightarrow{D} N(0, V(Y)) \text{ as } n\to\infty.$$

If $X_n \overset{D}{=} \sum_{i=1}^{n} Y_i$ where the Y_i are iid, then

$$\sqrt{n}\left(\frac{X_n}{n} - E(Y)\right) \overset{D}{=} \sqrt{n}\left(\bar{Y}_n - E(Y)\right) \xrightarrow{D} N(0, V(Y)).$$

If formula for $E[Y^k]$ is given,

$$E(Y) = E(Y^1), \text{ and } V(Y) = E(Y^2) - [E(Y)]^2$$

III) $X_n \xrightarrow{D} X$ if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ at all continuity points x of $F_X(x)$. ← cdf

If $\lim_{n\to\infty} F_{X_n}(x) = F(x)$ where $F(x)$ is continuous, need $F(x)$ to be a cdf. Hence if $F(x) \equiv c$ $\forall x$, then X_n does not converge in distribution to X.

If $\lim_{n\to\infty} F_{X_n}(x) = F(x)$ except at some discontinuity points of $F(x)$,

need $F(x) = F_X(x)$ except at discontinuity points of cdf $F_X(x)$.

If $X_n \sim U(a_n, b_n)$, $F_{X_n}(x) = \underbrace{\frac{-a_n}{b_n - a_n}}_{c_n} + \frac{1}{b_n - a_n} t$

where $F_{X_n}(b_n) = c_n + \frac{b_n}{b_n - a_n} = 1$ and

$$F_{X_n}(a_n) = c_n + \frac{a_n}{b_n - a_n} = 0 \quad \text{so } c_n = \frac{-a_n}{b_n - a_n}.$$

iv) I like proofs for MCT, LDCT and b) of Quiz 6 #3.

v) suppose $f_n \geq 0$.
If $\lim_n f_n = f$ and $\lim_n \int f_n \, d\mu = c$

then
$$\begin{array}{ccc}
\int \liminf f_n \, d\mu & \leq & \liminf \int f_n \, d\mu \\
\| & & \| \\
\int \limsup f_n \, d\mu & \leq & \limsup \int f_n \, d\mu \\
\| & & \| \\
\int \lim f_n \, d\mu & \leq & \lim \int f_n \, d\mu = c \\
\| & & \\
\int f \, d\mu & \leq & \lim \int f_n \, d\mu = c
\end{array}$$

by Fatou's lemma.

If $\lim_n X_n = X$ and $\lim_n E(X_n) = c$

with $X_n \geq 0$ then

$$E[\liminf X_n] \leq \liminf E(X_n)$$
$$\| \qquad\qquad \|$$
$$E[\limsup X_n] \leq \limsup E(X_n)$$
$$\| \qquad\qquad \|$$
$$E(\lim X_n) \leq \lim E(X_n) = c$$
$$\| \qquad\qquad \|$$
$$E(X) \leq \lim E(X_n) = c$$ by Fatou.

vi) $\int I_A \, d\mu = \mu(A)$ and $E[I_A] = \int I_A \, dP = P(A)$

for event A.

vii) The induced prob $P_{\underset{\sim}{X}}(B) = P(\underset{\sim}{X}^{-1}(B))$

$\forall \, B \in \mathcal{B}(\mathbb{R}^k)$.

viii) 72) and 73) from E2 rev are useful for measurable functions. See reasonable proofs notes 46 v) ⇒ vi), vi) ⇒ v)

notes 47 b), c)

notes 48 4)].

xi) Fubini's th allows double integrals to be computed with iterated integrals E2 rev 84), 85), 87).