Math 582   HW 10, 2024 Due Wednesday, Nov. 13.
Final: TBAday, : time TBA

This homework has 2 pages, 5 problems.

1) **Olive 6.3.** For ridge regression, let $\boldsymbol{A}_n = (\boldsymbol{W}^T\boldsymbol{W} + \lambda_{1,n}\boldsymbol{I}_{p-1})^{-1}\boldsymbol{W}^T\boldsymbol{W}$ and $\boldsymbol{B}_n = [\boldsymbol{I}_{p-1} - \lambda_{1,n}(\boldsymbol{W}^T\boldsymbol{W} + \lambda_{1,n}\boldsymbol{I}_{p-1})^{-1}]$. Show $\boldsymbol{A}_n - \boldsymbol{B}_n = \boldsymbol{0}$.

2) Suppose that $Y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i,5} + e_i$.

a) Testing $H_0 : \beta_2 = \beta_4 = \beta_5 = 0$ is equivalent to testing $H_0 : \boldsymbol{A\beta} = \boldsymbol{0}$. What is $\boldsymbol{A}$? Hint: want $\boldsymbol{A\beta} = (\beta_2, \beta_4, \beta_5)^T$.

b) Testing $H_0 : \beta_2 = \beta_4 = \beta_5$ is equivalent to testing $H_0 : \boldsymbol{A\beta} = \boldsymbol{0}$. What is $\boldsymbol{A}$? Hint: want, for example, $\boldsymbol{A\beta} = (\beta_2 - \beta_4, \beta_2 - \beta_5)^T$.

3) Suppose $\boldsymbol{x}_1, ..., \boldsymbol{x}_n$ are iid 7×1 random vectors where $E(\boldsymbol{x}_i) = \boldsymbol{\mu}$ and $\text{Cov}(\boldsymbol{x}_i) = \sum_j \pi_j \boldsymbol{\Sigma}_j$. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{x}} - \boldsymbol{d})$ for appropriate vector $\boldsymbol{d}$.

4) The linear model is $\boldsymbol{Z} = \boldsymbol{W\eta} + \boldsymbol{e}$.

Theorem: Assume that the sample correlation matrix $\boldsymbol{R}_{\boldsymbol{u}} = \dfrac{\boldsymbol{W}^T\boldsymbol{W}}{n} \xrightarrow{P} \boldsymbol{V}^{-1}$. Let $\boldsymbol{H} = \boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{W}^T = (h_{ij})$, and assume that $\max_{i=1,...,n} h_{ii} \xrightarrow{P} 0$ as $n \to \infty$. Let $\hat{\boldsymbol{\eta}}_A$ be $\hat{\boldsymbol{\eta}}_{EN}$, $\hat{\boldsymbol{\eta}}_L$, or $\hat{\boldsymbol{\eta}}_R$. Let p be fixed.

i) OLS CLT: $\sqrt{n}(\hat{\boldsymbol{\eta}}_{OLS} - \boldsymbol{\eta}) \xrightarrow{D} N_{p-1}(\boldsymbol{0}, \sigma^2\boldsymbol{V})$.

ii) *If* $\hat{\lambda}_{1,n}/\sqrt{n} \xrightarrow{P} 0$, *then*

$$\sqrt{n}(\hat{\boldsymbol{\eta}}_A - \boldsymbol{\eta}) \xrightarrow{D} N_{p-1}(\boldsymbol{0}, \sigma^2\boldsymbol{V}).$$

iii) *If* $\hat{\lambda}_{1,n}/\sqrt{n} \xrightarrow{P} \tau \geq 0$, $\hat{\alpha} \xrightarrow{P} \psi \in [0,1]$, *and* $\boldsymbol{s}_n \xrightarrow{P} \boldsymbol{s} = \boldsymbol{s}_{\boldsymbol{\eta}}$, *then*

$$\sqrt{n}(\hat{\boldsymbol{\eta}}_{EN} - \boldsymbol{\eta}) \xrightarrow{D} N_{p-1}\left(-\boldsymbol{V}[(1-\psi)\tau\boldsymbol{\eta} + \psi\tau\boldsymbol{s}], \sigma^2\boldsymbol{V}\right).$$

iv) *If* $\hat{\lambda}_{1,n}/\sqrt{n} \xrightarrow{P} \tau \geq 0$, *then*

$$\sqrt{n}(\hat{\boldsymbol{\eta}}_R - \boldsymbol{\eta}) \xrightarrow{D} N_{p-1}(-\tau\boldsymbol{V\eta}, \sigma^2\boldsymbol{V}).$$

v) *If* $\hat{\lambda}_{1,n}/\sqrt{n} \xrightarrow{P} \tau \geq 0$ *and* $\boldsymbol{s}_n \xrightarrow{P} \boldsymbol{s} = \boldsymbol{s}_{\boldsymbol{\eta}}$, *then*

$$\sqrt{n}(\hat{\boldsymbol{\eta}}_L - \boldsymbol{\eta}) \xrightarrow{D} N_{p-1}\left(\frac{-\tau}{2}\boldsymbol{Vs}, \sigma^2\boldsymbol{V}\right).$$

ii) and v) are the Lasso CLT, ii) and iv) are the RR CLT, and ii) and iii) are the EN CLT.

a) Which estimator (OLS, Lasso L, ridge regression RR, or elastic net EN) has the best large sample theory?

b) Let $\boldsymbol{A}$ be a $k \times (p-1)$ constant matrix. If $\hat{\lambda}_{1,n}/\sqrt{n} \xrightarrow{P} 0$, then

$$\sqrt{n}\boldsymbol{A}(\hat{\boldsymbol{\eta}}_A - \boldsymbol{\eta}) \xrightarrow{D} \boldsymbol{z}.$$

Find the distribution of $\boldsymbol{z}$.

5) Let $\boldsymbol{\Sigma}_i$ be the nonsingular population covariance matrix of the ith treatment group or population. To simplify the large sample theory, assume $n_i = \pi_i n$ where $0 < \pi_i < 1$ and $\sum_{i=1}^{2} \pi_i = 1$. Let T_i be a multivariate location estimator such that $\sqrt{n_i}(T_i - \boldsymbol{\mu}_i) \xrightarrow{D} N_m(\boldsymbol{0}, \boldsymbol{\Sigma}_i)$, and $\sqrt{n}(T_i - \boldsymbol{\mu}_i) \xrightarrow{D} N_m\left(\boldsymbol{0}, \frac{\boldsymbol{\Sigma}_i}{\pi_i}\right)$ for $i = 1, 2$. Assume $T_1 \perp\!\!\!\perp T_2$.

Then

$$\sqrt{n} \begin{bmatrix} T_1 - \boldsymbol{\mu}_1 \\ T_2 - \boldsymbol{\mu}_2 \end{bmatrix} \xrightarrow{D} \boldsymbol{u}.$$

Find the distribution of $\boldsymbol{u}$.