`Math 582    HW 3, 2024  Due Wednesday, Sept. 11.`

This homework has 2 pages, 6 problems.

1) Suppose X_n has cdf

$$F_n(x) = 1 - \left(1 - \frac{x}{\theta n}\right)^n$$

for $x \geq 0$ and $F_n(x) = 0$ for $x < 0$. Show that $X_n \xrightarrow{D} X$ by finding the cdf of X.

2) Suppose $Y_1, ..., Y_n$ are iid $EXP(\lambda)$. Then the MLE of λ is $\hat{\lambda}_n = \overline{Y}_n$, and $I_1(\lambda) = 1/\lambda^2$.

a) Find the limiting distribution of $\sqrt{n}(\overline{Y}_n - c)$ for appropriate constant c.

b) The Standard Limit Theorem for the MLE $\hat{\lambda}_n$ says

$$\sqrt{n}(\hat{\lambda}_n - \lambda) \xrightarrow{D} N\left(0, \frac{1}{I_1(\lambda)}\right).$$

Using a), prove that the Standard Limit Theorem holds for Y_i iid $EXP(\lambda)$.

3) Suppose $Y_1, ..., Y_n$ are iid and $W_i = t(Y_i)$ for a function t such that $E(W_i) = \mu_W$ and $V(W_i) = \sigma^2_W$. a) Find the limiting distribution of $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n t(Y_i) - c\right)$ for appropriate constant c.

Repeat a) if $W_i = t(Y_i) = Y_i^k$ for positive integer k, assuming that $E(W_i)$ and $E(W_i^2)$ are finite. This part gives a limit theorem for the sample kth moment. So give simple formulas for c, μ_W, and σ^2_W.

4) (O2.22): Suppose that $Y_1, ..., Y_n$ are iid from a distribution with pdf $f(y|\theta)$ and that the integral and differentiation operators of all orders can be interchanged (e.g. the data is from a one parameter exponential family).

a) Show that $0 = E\left[\frac{\partial}{\partial\theta}\log(f(Y|\theta))\right]$ by showing that

$$\frac{\partial}{\partial\theta}1 = 0 = \frac{\partial}{\partial\theta}\int f(y|\theta)dy =$$

$$0 = \frac{\partial}{\partial\theta}\int f(y|\theta)dy = \int \left[\frac{\partial}{\partial\theta}\log(f(y|\theta))\right] f(y|\theta)dy. \tag{1}$$

b) Take 2nd derivatives of (1) to show that

$$I_1(\theta) = E_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(Y|\theta)\right)^2\right] = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log(f(Y|\theta))\right].$$

Hint:

$$I_1(\theta) = E_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(Y|\theta)\right)^2\right]$$

and a) and b) are basically the proof of Theorem 1.33b) in the Olive online text (p. 36).

5) (O2.24): Let $W_n = X_n - X$ and let $r > 0$. Notice that for any $\epsilon > 0$,

$$E|X_n - X|^r \geq E[|X_n - X|^r \ I(|X_n - X| \geq \epsilon)] \geq \epsilon^r P(|X_n - X| \geq \epsilon).$$

Show that $W_n \xrightarrow{P} 0$ if $E|X_n - X|^r \to 0$ as $n \to \infty$.

6) (O2.49): Let $P(X_n = 0) = 1 - 1/n^r$ and $P(X_n = n) = 1/n^r$ where $r > 0$.
a) Prove that X_n does not converge in rth mean to 0. Hint: Find $E[|X_n|^r]$.

b) Does $X_n \xrightarrow{D} X$ for some random variable X? Prove or disprove.