

Math 582 HW 4, 2024 Due Wednesday, Sept. 25.  
Exam 1 is on Wednesday, Sept. 18.

**This homework has 2 pages, 5 problems.**

1) Let  $Y_1, \dots, Y_n$  be iid,  $T_{1,n} = \bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$  and let  $T_{2,n} = \text{MED}(n)$  be the sample median. Let  $\theta = \mu = E(Y) = \text{MED}(Y)$ .

a) Find  $ARE(T_{1,n}, T_{2,n})$  if  $F$  is the cdf of the normal  $N(\mu, \sigma^2)$  distribution.

b) Find  $ARE(T_{1,n}, T_{2,n})$  if  $F$  is the cdf of the double exponential  $DE(\mu, \lambda)$  distribution which has pdf

$$f(y) = \frac{1}{2\lambda} \exp\left(\frac{-|y - \mu|}{\lambda}\right)$$

where  $y$  is real and  $\lambda > 0$ . Then  $E(Y) = \text{MED}(Y) = \mu$  and  $V(Y) = 2\lambda^2$ .

Hint: a) was done in class.

2) Let  $X_n$  be sequence of random variables with cdfs  $F_n$  and mgfs  $m_n$ . Let  $X$  be a random variable with cdf  $F$  and mgf  $m$ . Let

$$m_n(t) = \frac{1}{[1 - (\lambda + \frac{1}{n})t]}$$

for  $t < 1/(\lambda + 1/n)$ . Show  $m_n(t) \rightarrow m(t)$  for  $t < 1/\lambda$  and then use the continuity theorem to show  $X_n \xrightarrow{D} X$ .

3) Let  $X_1, \dots, X_n$  be independent identically distributed random variables with probability density function

$$f(x) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0.$$

Then  $E(X) = \theta/(\theta + 1)$ ,  $I_1(\theta) = 1/\theta^2$ , and

$$V(X) = \frac{\theta}{(\theta + 1)^2(\theta + 2)}.$$

Then  $W_i = -\log(X_i) \sim \text{EXP}(1/\theta)$ .

a) The MLE of  $1/\theta$  is

$$T_{1n} = \bar{W}_n = \frac{-\sum_{i=1}^n \log(X_i)}{n}.$$

Find the limiting distribution of  $\sqrt{n}(\bar{W}_n - \frac{1}{\theta})$ .

b) An alternative estimator of  $1/\theta$  is

$$T_{2n} = g(\bar{X}_n) = \frac{1 - \bar{X}_n}{\bar{X}_n}.$$

Find the limiting distribution of  $\sqrt{n}\left(g(\bar{X}_n) - \frac{1}{\theta}\right)$ .

c) Find  $ARE(T_{1n}, T_{2n})$ .

4) Suppose  $X_1, \dots, X_n$  are iid  $C(\mu, \sigma)$  with characteristic function  $c_X(t) = \exp(it\mu - |t|\sigma)$  where  $\exp(a) = e^a$ .

a) Find the characteristic function  $c_{T_n}(t)$  of  $T_n = \sum_{i=1}^n X_i$ .

b) Find the characteristic function of  $\bar{X}_n = T_n/n$ .

c) Does  $\bar{X}_n \xrightarrow{D} W$  for some RV  $W$ ? Explain.

5) In homework 3, problem 2, you had  $Y_1, \dots, Y_n$  are iid  $EXP(\lambda)$ . Then the MLE of  $\lambda$  is  $\hat{\lambda}_n = \bar{Y}_n$ , and  $I_1(\lambda) = 1/\lambda^2$ .

You showed

$$\sqrt{n}(\bar{Y}_n - \lambda) = \sqrt{n}(\hat{\lambda}_n - \lambda) \xrightarrow{D} N\left(0, \frac{1}{I_1(\lambda)}\right).$$

Is  $\hat{\lambda}_n = \bar{Y}_n$  asymptotically efficient?