Math 582   HW 4, 2022 Due Tuesday, Feb. 15.
Exam 1 is on Thursday, Feb. 11.

This homework has 2 pages, 5 problems.

1) Let $Y_1, ..., Y_n$ be iid, $T_{1,n} = \overline{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$ and let $T_{2,n} = \text{MED}(n)$ be the sample median. Let $\theta = \mu = E(Y) = MED(Y)$.

a) Find $ARE(T_{1,n}, T_{2,n})$ if F is the cdf of the normal $N(\mu, \sigma^2)$ distribution.

b) Find $ARE(T_{1,n}, T_{2,n})$ if F is the cdf of the double exponential $DE(\mu, \lambda)$ distribution which has pdf

$$f(y) = \frac{1}{2\lambda} \exp\left(\frac{-|y-\mu|}{\lambda}\right)$$

where y is real and $\lambda > 0$. Then $E(Y) = MED(Y) = \mu$ and $V(Y) = 2\lambda^2$.

Hint: a) was done in class.

2) Let X_n be sequence of random variables with cdfs F_n and mgfs m_n. Let X be a random variable with cdf F and mgf m. Let

$$m_n(t) = \frac{1}{[1 - (\lambda + \frac{1}{n})t]}$$

for $t < 1/(\lambda + 1/n)$. Show $m_n(t) \to m(t)$ for $t < 1/\lambda$ and then use the continuity theorem to show $X_n \xrightarrow{D} X$.

3) Let $X_1, ..., X_n$ be independent identically distributed random variables with probability density function

$$f(x) = \theta x^{\theta - 1}, \;\; 0 < x < 1, \;\; \theta > 0.$$

Then $E(X) = \theta/(\theta+1)$, $I_1(\theta) = 1/\theta^2$, and

$$V(X) = \frac{\theta}{(\theta+1)^2(\theta+2)}.$$

Then $W_i = -\log(X_i) \sim EXP(1/\theta)$.

a) The MLE of $1/\theta$ is

$$T_{1n} = \overline{W}_n = \frac{-\sum_{i=1}^n \log(X_i)}{n}.$$

Find the limiting distribution of $\sqrt{n}(\overline{W}_n - \frac{1}{\theta})$.

b) An alternative estimator of $1/\theta$ is

$$T_{2n} = g(\overline{X}_n) = \frac{1 - \overline{X}_n}{\overline{X}_n}.$$

Find the limiting distribution of $\sqrt{n}\left(g(\overline{X}_n) - \frac{1}{\theta}\right)$.

c) Find $ARE(T_{1n}, T_{2n})$.

4) Suppose $X_1, ..., X_n$ are iid $C(\mu, \sigma)$ with characteristic function $c_X(t) = \exp(it\mu - |t|\sigma)$ where $\exp(a) = e^a$.

a) Find the characteristic function $c_{T_n}(t)$ of $T_n = \sum_{i=1}^n X_i$.

b) Find the characteristic function of $\overline{X}_n = T_n/n$.

c) Does $\overline{X}_n \xrightarrow{D} W$ for some RV W? Explain.

5) In homework 3, problem 2, you had $Y_1, ..., Y_n$ are iid $EXP(\lambda)$. Then the MLE of λ is $\hat{\lambda}_n = \overline{Y}_n$, and $I_1(\lambda) = 1/\lambda^2$.

You showed

$$\sqrt{n}(\overline{Y}_n - \lambda) = \sqrt{n}(\hat{\lambda}_n - \lambda) \xrightarrow{D} N\left(0, \frac{1}{I_1(\lambda)}\right).$$

Is $\hat{\lambda}_n = \overline{Y}_n$ asymptotically efficient?