`Math 582    HW 5, 2022  Due Tuesday, Feb. 22.`

This homework has 1 page, 6 problems.

1) Suppose $X_1, ..., X_n$ are iid from a distribution with mean μ and variance σ^2. The method of moments estimator for σ^2 is

$$S^2_M = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X}_n)^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 \quad - \quad (\overline{X}_n)^2.$$

a) $\displaystyle\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{P} c$. What is c? Hint: Use WLLN on $W_i = X_i^2$.

b) $(\overline{X}_n)^2 \xrightarrow{P} d$. What is d? Hint: $g(x) = x^2$ is continuous, so if $Z_n \xrightarrow{P} \theta$, then $g(Z_n) \xrightarrow{P} g(\theta)$.

c) Show $S^2_m \xrightarrow{P} \sigma^2$.

d) $\displaystyle S^2 = \frac{n}{n-1} S^2_M = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$. Prove $S^2 \xrightarrow{P} \sigma^2$.

2) Suppose $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ are iid $k \times 1$ random vectors where $E(\boldsymbol{X}_i) = (\mu_1, ..., \mu_k)^T$ and $Cov(\boldsymbol{X}_i) = (1-\alpha)\boldsymbol{I} + \alpha\mathbf{1}\mathbf{1}^T$, where $\boldsymbol{I}$ is the $k \times k$ identity matrix, $\mathbf{1} = (1, 1, ..., 1)^T$, and $-(k-1)^{-1} < \alpha < 1$. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{X}} - \boldsymbol{c})$ for appropriate vector $\boldsymbol{c}$.

3) Suppose X_n are random variables with characteristic functions $c_{X_n}(t)$, and that $c_{X_n}(t) \to e^{itc}$ for every $t \in \mathbb{R}$ where c is a constant. Does $X_n \xrightarrow{D} X$ for some random variable X? Explain briefly. Hint: Is the function $g(t) = e^{itc}$ continuous as $t = 0$? Is there a random variable that has characteristic function $g(t)$?

4) The characteristic function for $Y \sim N(\mu, \sigma^2)$ is $c_Y(t) = \exp(it\mu - t^2\sigma^2/2)$. Let $X_n \sim N(0, n)$.

a) Prove $c_{X_n}(t) \to h(t)\ \forall t$ by finding $h(t)$.

b) Use a) to prove whether X_n converges in distribution. Hint: Exam 2 review 36) may be useful.

5) Suppose

$$Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \xrightarrow{D} N(0,1)$$

and $s_n^2 \xrightarrow{P} \sigma^2$ where $\sigma > 0$. Prove that

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{s_n} \xrightarrow{D} N(0,1).$$

6) If $Y_n \xrightarrow{D} Y$, $a_n \xrightarrow{P} a$, and $b_n \xrightarrow{P} b$, then $a_n + b_n Y_n \xrightarrow{D} X$. Find X.