

**This homework has 1 page, 6 problems.**

1) Suppose  $X_1, \dots, X_n$  are iid from a distribution with mean  $\mu$  and variance  $\sigma^2$ . The method of moments estimator for  $\sigma^2$  is

$$S_M^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n} \sum_{i=1}^n X_i^2 - (\bar{X}_n)^2.$$

- a)  $\frac{1}{n} \sum_{i=1}^n X_i^2 \xrightarrow{P} c$ . What is  $c$ ? Hint: Use WLLN on  $W_i = X_i^2$ .
- b)  $(\bar{X}_n)^2 \xrightarrow{P} d$ . What is  $d$ ? Hint:  $g(x) = x^2$  is continuous, so if  $Z_n \xrightarrow{P} \theta$ , then  $g(Z_n) \xrightarrow{P} g(\theta)$ .
- c) Show  $S_m^2 \xrightarrow{P} \sigma^2$ .
- d)  $S^2 = \frac{n}{n-1} S_M^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ . Prove  $S^2 \xrightarrow{P} \sigma^2$ .

2) Suppose  $\mathbf{X}_1, \dots, \mathbf{X}_n$  are iid  $k \times 1$  random vectors where  $E(\mathbf{X}_i) = (\mu_1, \dots, \mu_k)^T$  and  $Cov(\mathbf{X}_i) = (1 - \alpha)\mathbf{I} + \alpha\mathbf{1}\mathbf{1}^T$ , where  $\mathbf{I}$  is the  $k \times k$  identity matrix,  $\mathbf{1} = (1, 1, \dots, 1)^T$ , and  $-(k-1)^{-1} < \alpha < 1$ . Find the limiting distribution of  $\sqrt{n}(\bar{\mathbf{X}} - \mathbf{c})$  for appropriate vector  $\mathbf{c}$ .

3) Suppose  $X_n$  are random variables with characteristic functions  $c_{X_n}(t)$ , and that  $c_{X_n}(t) \rightarrow e^{itc}$  for every  $t \in \mathbb{R}$  where  $c$  is a constant. Does  $X_n \xrightarrow{D} X$  for some random variable  $X$ ? Explain briefly. Hint: Is the function  $g(t) = e^{itc}$  continuous as  $t = 0$ ? Is there a random variable that has characteristic function  $g(t)$ ?

4) The characteristic function for  $Y \sim N(\mu, \sigma^2)$  is  $c_Y(t) = \exp(it\mu - t^2\sigma^2/2)$ . Let  $X_n \sim N(0, n)$ .

- a) Prove  $c_{X_n}(t) \rightarrow h(t) \forall t$  by finding  $h(t)$ .
- b) Use a) to prove whether  $X_n$  converges in distribution. Hint: Exam 2 review 36) may be useful.
- 5) Suppose

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{D} N(0, 1)$$

and  $s_n^2 \xrightarrow{P} \sigma^2$  where  $\sigma > 0$ . Prove that

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{s_n} \xrightarrow{D} N(0, 1).$$

- 6) If  $Y_n \xrightarrow{D} Y$ ,  $a_n \xrightarrow{P} a$ , and  $b_n \xrightarrow{P} b$ , then  $a_n + b_n Y_n \xrightarrow{D} X$ . Find  $X$ .