

**This homework has 2 pages, 4 problems.**

1) Suppose

$$\sqrt{n} \left( \begin{pmatrix} \hat{\lambda} \\ \hat{\boldsymbol{\eta}} \end{pmatrix} - \begin{pmatrix} \lambda \\ \boldsymbol{\eta} \end{pmatrix} \right) \xrightarrow{D} N_{p+1} \left( \begin{pmatrix} 0 \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \Sigma_\lambda & \Sigma_{\lambda\boldsymbol{\eta}} \\ \Sigma_{\boldsymbol{\eta}\lambda} & \Sigma_{\boldsymbol{\eta}} \end{pmatrix} \right) \sim N_{p+1}(\mathbf{0}, \boldsymbol{\Sigma})$$

where  $\lambda$  is a scalar and  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_p)$ . Let

$$\mathbf{g} \begin{pmatrix} \lambda \\ \boldsymbol{\eta} \end{pmatrix} = \lambda \boldsymbol{\eta} =$$

$(\lambda\eta_1, \dots, \lambda\eta_p)^T$ . Then

$$\sqrt{n}(\hat{\lambda}\hat{\boldsymbol{\eta}} - \lambda\boldsymbol{\eta}) \xrightarrow{D} N_p \left( \mathbf{0}, \mathbf{D}_{\mathbf{g}(\boldsymbol{\theta})} \boldsymbol{\Sigma} \mathbf{D}_{\mathbf{g}(\boldsymbol{\theta})}^T \right)$$

by the Multivariate Delta Method.

a) Find  $\mathbf{D}_{\mathbf{g}(\boldsymbol{\theta})}$ .

b) Let  $\mathbf{A}$  be a  $k \times p$  full rank constant matrix with  $k \leq p$  and  $\mathbf{0} = \mathbf{A}\boldsymbol{\eta}$ . Find  $\mathbf{A}\mathbf{D}_{\mathbf{g}(\boldsymbol{\theta})}$ .

Note: then  $\sqrt{n}(\mathbf{A}\hat{\boldsymbol{\eta}} - \mathbf{0}) \xrightarrow{D} N_p \left( \mathbf{0}, \mathbf{A}\mathbf{D}_{\mathbf{g}(\boldsymbol{\theta})} \boldsymbol{\Sigma} \mathbf{D}_{\mathbf{g}(\boldsymbol{\theta})}^T \mathbf{A}^T \right)$ .

2) Suppose

$$\sqrt{n} \left( \begin{pmatrix} \hat{\sigma}_1^2 \\ \vdots \\ \hat{\sigma}_p^2 \end{pmatrix} - \begin{pmatrix} \sigma_1^2 \\ \vdots \\ \sigma_p^2 \end{pmatrix} \right) \xrightarrow{D} N_p(\mathbf{0}, \boldsymbol{\Sigma}).$$

Let  $\boldsymbol{\theta} = (\sigma_1^2, \dots, \sigma_p^2)^T$  and let  $\mathbf{g}(\boldsymbol{\theta}) = (\log(\sigma_1^2), \dots, \log(\sigma_p^2))^T$ . Find  $\mathbf{D}_{\mathbf{g}(\boldsymbol{\theta})}$ .

3) It is true that  $W_n$  has the same order as  $X_n$  in probability, written  $W_n \asymp_P X_n$ , iff for every  $\epsilon > 0$  there exist positive constants  $N_\epsilon$  and  $0 < d_\epsilon < D_\epsilon$  such that

$$P(d_\epsilon \leq \left| \frac{W_n}{X_n} \right| \leq D_\epsilon) \geq 1 - \epsilon$$

for all  $n \geq N_\epsilon$ .

a) Show that if  $W_n \asymp_P X_n$  then  $X_n \asymp_P W_n$ .

b) Show that if  $W_n \asymp_P X_n$  then  $W_n = O_P(X_n)$ .

c) Show that if  $W_n \asymp_P X_n$  then  $X_n = O_P(W_n)$ .

d) Show that if  $W_n = O_P(X_n)$  and if  $X_n = O_P(W_n)$ , then  $W_n \asymp_P X_n$ .

Hint: See Theorem 2.28 on page 72 of the Olive notes.

4) This problem will prove the following Theorem which says that if there are  $K$  estimators  $T_{j,n}$  of a parameter  $\beta$ , such that  $\|T_{j,n} - \beta\| = O_P(n^{-\delta})$  where  $0 < \delta \leq 1$ , and if  $T_n^*$  picks one of these estimators, then  $\|T_n^* - \beta\| = O_P(n^{-\delta})$ .

**Lemma: Pratt (1959).** Let  $X_{1,n}, \dots, X_{K,n}$  each be  $O_P(1)$  where  $K$  is fixed. Suppose  $W_n = X_{i_n,n}$  for some  $i_n \in \{1, \dots, K\}$ . Then

$$W_n = O_P(1). \quad (1)$$

**Proof.**

$$P(\max\{X_{1,n}, \dots, X_{K,n}\} \leq x) = P(X_{1,n} \leq x, \dots, X_{K,n} \leq x) \leq$$

$$F_{W_n}(x) \leq P(\min\{X_{1,n}, \dots, X_{K,n}\} \leq x) = 1 - P(X_{1,n} > x, \dots, X_{K,n} > x).$$

Since  $K$  is finite, there exists  $B > 0$  and  $N$  such that  $P(X_{i,n} \leq B) > 1 - \epsilon/2K$  and  $P(X_{i,n} > -B) > 1 - \epsilon/2K$  for all  $n > N$  and  $i = 1, \dots, K$ . Bonferroni's inequality states that  $P(\cap_{i=1}^K A_i) \geq \sum_{i=1}^K P(A_i) - (K - 1)$ . Thus

$$F_{W_n}(B) \geq P(X_{1,n} \leq B, \dots, X_{K,n} \leq B) \geq K(1 - \epsilon/2K) - (K - 1) = K - \epsilon/2 - K + 1 = 1 - \epsilon/2$$

and

$$\begin{aligned} -F_{W_n}(-B) &\geq -1 + P(X_{1,n} > -B, \dots, X_{K,n} > -B) \geq \\ &-1 + K(1 - \epsilon/2K) - (K - 1) = -1 + K - \epsilon/2 - K + 1 = -\epsilon/2. \end{aligned}$$

Hence

$$F_{W_n}(B) - F_{W_n}(-B) \geq 1 - \epsilon \text{ for } n > N. \text{ QED}$$

**Theorem.** Suppose  $\|T_{j,n} - \beta\| = O_P(n^{-\delta})$  for  $j = 1, \dots, K$  where  $0 < \delta \leq 1$ . Let  $T_n^* = T_{i_n,n}$  for some  $i_n \in \{1, \dots, K\}$  where, for example,  $T_{i_n,n}$  is the  $T_{j,n}$  that minimized some criterion function. Then

$$\|T_n^* - \beta\| = O_P(n^{-\delta}). \quad (2)$$

Prove the above theorem using the Lemma with an appropriate  $X_{j,n}$ .

Hint: See Theorems 2.29 and 2.30 in the Olive notes.