Math 582 HW 7, 2024 Due Wednesday, Oct. 16.

This homework has 2 pages, 4 problems.

1) Suppose

$$\sqrt{n}\left(\begin{pmatrix} \hat{\lambda} \\ \hat{\boldsymbol{\eta}} \end{pmatrix} - \begin{pmatrix} \lambda \\ \boldsymbol{\eta} \end{pmatrix} \right) \xrightarrow{D} N_{p+1}\left(\begin{pmatrix} 0 \\ \boldsymbol{0} \end{pmatrix}, \begin{pmatrix} \Sigma_\lambda & \boldsymbol{\Sigma}_{\lambda\boldsymbol{\eta}} \\ \boldsymbol{\Sigma}_{\boldsymbol{\eta}\lambda} & \boldsymbol{\Sigma}_{\boldsymbol{\eta}} \end{pmatrix} \right) \sim N_{p+1}(\boldsymbol{0}, \boldsymbol{\Sigma})$$

where λ is a scalar and $\boldsymbol{\eta} = (\eta_1, ..., \eta_p)$. Let

$$\boldsymbol{g}\begin{pmatrix} \lambda \\ \boldsymbol{\eta} \end{pmatrix} = \lambda\boldsymbol{\eta} =$$

$(\lambda\eta_1, ..., \lambda\eta_p)^T$. Then

$$\sqrt{n}(\hat{\lambda}\hat{\boldsymbol{\eta}} - \lambda\boldsymbol{\eta}) \xrightarrow{D} N_p\left(\boldsymbol{0}, \boldsymbol{D}_{\boldsymbol{g}(\boldsymbol{\theta})}\boldsymbol{\Sigma}\boldsymbol{D}^T_{\boldsymbol{g}(\boldsymbol{\theta})}\right)$$

by the Multivariate Delta Method.

a) Find $\boldsymbol{D}_{\boldsymbol{g}(\boldsymbol{\theta})}$.

b) Let $\boldsymbol{A}$ be a $k \times p$ full rank constant matrix with $k \leq p$ and $\boldsymbol{0} = \boldsymbol{A}\boldsymbol{\eta}$. Find $\boldsymbol{A}\boldsymbol{D}_{\boldsymbol{g}(\boldsymbol{\theta})}$.

Note: then $\sqrt{n}(\boldsymbol{A}\hat{\lambda}\hat{\boldsymbol{\eta}} - \boldsymbol{0}) \xrightarrow{D} N_p\left(\boldsymbol{0}, \boldsymbol{A}\boldsymbol{D}_{\boldsymbol{g}(\boldsymbol{\theta})}\boldsymbol{\Sigma}\boldsymbol{D}^T_{\boldsymbol{g}(\boldsymbol{\theta})}\boldsymbol{A}^T\right)$.

2) Suppose

$$\sqrt{n}\left(\begin{pmatrix} \hat{\sigma}_1^2 \\ \vdots \\ \hat{\sigma}_p^2 \end{pmatrix} - \begin{pmatrix} \sigma_1^2 \\ \vdots \\ \sigma_p^2 \end{pmatrix} \right) \xrightarrow{D} N_p(\boldsymbol{0}, \boldsymbol{\Sigma}).$$

Let $\boldsymbol{\theta} = (\sigma_1^2, ..., \sigma_p^2)^T$ and let $\boldsymbol{g}(\boldsymbol{\theta}) = (\log(\sigma_1^2), ..., \log(\sigma_p^2))^T$. Find $\boldsymbol{D}_{\boldsymbol{g}(\boldsymbol{\theta})}$.

3) It is true that W_n has the same order as X_n in probability, written $W_n \asymp_P X_n$, iff for every $\epsilon > 0$ there exist positive constants N_ϵ and $0 < d_\epsilon < D_\epsilon$ such that

$$P(d_\epsilon \leq \left|\frac{W_n}{X_n}\right| \leq D_\epsilon) \geq 1 - \epsilon$$

for all $n \geq N_\epsilon$.

a) Show that if $W_n \asymp_P X_n$ then $X_n \asymp_P W_n$.

b) Show that if $W_n \asymp_P X_n$ then $W_n = O_P(X_n)$.

c) Show that if $W_n \asymp_P X_n$ then $X_n = O_P(W_n)$.

d) Show that if $W_n = O_P(X_n)$ and if $X_n = O_P(W_n)$, then $W_n \asymp_P X_n$.

Hint: See Theorem 2.28 in the Olive notes.

4) This problem will prove the following Theorem which says that if there are K estimators $T_{j,n}$ of a parameter $\boldsymbol{\beta}$, such that $\|T_{j,n} - \boldsymbol{\beta}\| = O_P(n^{-\delta})$ where $0 < \delta \leq 1$, and if T_n^* picks one of these estimators, then $\|T_n^* - \boldsymbol{\beta}\| = O_P(n^{-\delta})$.

Lemma: Pratt (1959). Let $X_{1,n}, ..., X_{K,n}$ each be $O_P(1)$ where K is fixed. Suppose $W_n = X_{i_n,n}$ for some $i_n \in \{1, ..., K\}$. Then

$$W_n = O_P(1). \tag{1}$$

Proof.

$$P(\max\{X_{1,n}, ..., X_{K,n}\} \leq x) = P(X_{1,n} \leq x, ..., X_{K,n} \leq x) \leq$$

$$F_{W_n}(x) \leq P(\min\{X_{1,n}, ..., X_{K,n}\} \leq x) = 1 - P(X_{1,n} > x, ..., X_{K,n} > x).$$

Since K is finite, there exists $B > 0$ and N such that $P(X_{i,n} \leq B) > 1 - \epsilon/2K$ and $P(X_{i,n} > -B) > 1 - \epsilon/2K$ for all $n > N$ and $i = 1, ..., K$. Bonferroni's inequality states that $P(\cap_{i=1}^K A_i) \geq \sum_{i=1}^K P(A_i) \ - \ (K-1)$. Thus

$$F_{W_n}(B) \geq P(X_{1,n} \leq B, ..., X_{K,n} \leq B) \geq K(1-\epsilon/2K)-(K-1) = K-\epsilon/2-K+1 = 1-\epsilon/2$$

and

$$-F_{W_n}(-B) \geq -1 + P(X_{1,n} > -B, ..., X_{K,n} > -B) \geq$$

$$-1 + K(1 - \epsilon/2K) - (K-1) = -1 + K - \epsilon/2 - K + 1 = -\epsilon/2.$$

Hence

$$F_{W_n}(B) - F_{W_n}(-B) \geq 1 - \epsilon \ \text{ for } \ \text{n} > \text{N}. \ \text{ QED}$$

Theorem. Suppose $\|T_{j,n} - \boldsymbol{\beta}\| = O_P(n^{-\delta})$ for $j = 1, ..., K$ where $0 < \delta \leq 1$. Let $T_n^* = T_{i_n,n}$ for some $i_n \in \{1, ..., K\}$ where, for example, $T_{i_n,n}$ is the $T_{j,n}$ that minimized some criterion function. Then

$$\|T_n^* - \boldsymbol{\beta}\| = O_P(n^{-\delta}). \tag{2}$$

Prove the above theorem using the Lemma with an appropriate $X_{j,n}$.

Hint: See Theorems 2.29 and 2.30 in the Olive notes.