`Math 582    HW 8, 2024  Due Wednesday, Oct. 30.`

This homework has 2 pages, 5 problems.

1) Suppose $\sqrt{n}(T_n - \theta) \xrightarrow{D} N(0, \sigma_T^2)$ and that $\hat{\sigma}_T^2$ is a consistent estimator of $\sigma_T^2 > 0$. Then a large sample 95.45% confidence interval for θ is $[T_n - 2SE(T_n), T_n + 2SE(T_n)]$ where $SE(T_n) = \hat{\sigma}_T/\sqrt{n}$. For the test $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$, fail to reject H_0 if θ_0 is in the CI, otherwise reject H_0. The power of the test $= P_\theta(\text{reject } H_0)$ which goes to 1 as $n \to \infty$ if $\theta \neq \theta_O$ because the length of the CI $\to 0$ as $n \to \infty$. The type I error = P(CI does not contain θ_0) when H_0 is true, and the type I error $\approx 1 - P(-2 < Z < 2) = 1 - 0.9544 = 0.0456$.

Consider the CI

$$[T_n - 2[\log_{10}(n)]^\gamma SE(T_n), T_n + 2[\log_{10}(n)]^\gamma SE(T_n)]$$

where γ is a number like 1/2, 1/3 or 1/4. Take $\gamma = 1/2$.

a) What does the power of the corresponding test converge to as $n \to \infty$?

b) What does the type I error converge to as $n \to \infty$?

c) For what value of n is $2\sqrt{\log_{10}(n)} = 4$?

2) Suppose $\sqrt{n}(T_n - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}$, $\sqrt{n}(T_i^* - T_n) \xrightarrow{D} \boldsymbol{u}$, and $\sqrt{n}(\overline{T}^* - T_n) \xrightarrow{P} \boldsymbol{0}$ where $E(\boldsymbol{u}) = \boldsymbol{0}$ and $Cov(\boldsymbol{u}) = \boldsymbol{\Sigma}_{\boldsymbol{u}} > 0$.

a) Prove $\sqrt{n}(\overline{T}^* - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}$.

b) $\sqrt{n}(T_i^* - \overline{T}^*) \xrightarrow{D} \boldsymbol{u}$.

Hint: add $a - a = 0$ to the term in parentheses for a good choice of a, and use Slutsky's Theorem.

3) Suppose $\sqrt{n}(T_n - \boldsymbol{\theta}) \xrightarrow{D} \boldsymbol{u}$ and $\boldsymbol{C}_n^{-1} \xrightarrow{P} \boldsymbol{C}^{-1}$. Then $n(T_n - \boldsymbol{\theta})^T \boldsymbol{C}_n^{-1}(T_n - \boldsymbol{\theta}) \xrightarrow{D} D^2$.

a) What is D^2 (e.g. is $D^2 = \boldsymbol{u}^T \boldsymbol{C} \boldsymbol{u}$)?

b) If $\boldsymbol{C}_n = \boldsymbol{I}_g$ for all positive integers n, what is D^2?

4) For a Poisson regression model, $Y|\boldsymbol{x} \sim \text{Poisson}[\exp(\boldsymbol{x}^T\boldsymbol{\beta})]$. Suppose $\hat{\boldsymbol{\beta}}_n$ is a $\sqrt{n}$ consistent estimator of $\boldsymbol{\beta}$. Let $W_n \sim \text{Poisson}[\exp(\boldsymbol{x}_f^T\hat{\boldsymbol{\beta}}_n)]$. Treat $\boldsymbol{x}_f$ as a known constant vector. Then W_n approximates W. What is the distribution of W?

(Note: to get a prediction interval for $Y_f|\boldsymbol{x}_f$, generate an iid sample $W_1, ..., W_B$ where $W_i \sim \text{Poisson}[\exp(\boldsymbol{x}_f^T\hat{\boldsymbol{\beta}}_n)]$. Then compute the shorth PI from the W_i. This technique is called the parametric bootstrap. It is not clear whether $W_n \xrightarrow{D} W$.)

5) The table on the next page shows simulation results for bootstrapping OLS (reg) and forward selection (vs) with C_p when $\boldsymbol{\beta} = (1, 1, 0, 0, 0)^T$. The β_i columns give coverage = the proportion of CIs that contained β_i and the average length of the CI. The test is for $H_0 : (\beta_3, \beta_4, \beta_5)^T = \boldsymbol{0}$ and H_0 is true. The "coverage" is the proportion of times the prediction region method bootstrap test failed to reject H_0. Since 1000 runs were used, a cov in [0.93,0.97] is reasonable for a nominal value of 0.95. Output is given for three different error distributions. If the coverage for both methods ≥ 0.93, the method with the shorter average CI length was more precise. (If one method had coverage ≥ 0.93 and

the other had coverage < 0.93, we will say the method with coverage ≥ 0.93 was more precise.)

a) For β_3, β_4, and β_5, which method, forward selection or the OLS full model, was more precise?

Table 1: Bootstrapping Forward Selection, $n = 100, p = 5, \psi = 0, B = 1000$

		β_1	β_2	β_3	β_4	β_5	test
reg	cov	0.95	0.93	0.93	0.93	0.94	0.93
	len	0.658	0.672	0.673	0.674	0.674	2.861
vs	cov	0.95	0.94	0.998	0.998	0.999	0.993
	len	0.661	0.679	0.546	0.548	0.544	3.11
reg	cov	0.96	0.93	0.94	0.96	0.93	0.94
	len	0.229	0.230	0.229	0.231	0.230	2.787
vs	cov	0.95	0.94	0.999	0.997	0.999	0.995
	len	0.228	0.229	0.185	0.187	0.186	3.056
reg	cov	0.94	0.94	0.95	0.94	0.94	0.93
	len	0.393	0.398	0.399	0.399	0.398	2.839
vs	cov	0.94	0.95	0.997	0.997	0.996	0.990
	len	0.392	0.400	0.320	0.322	0.321	3.077

b) The test "length" is the average length of the interval $[0, D_{(U_B)}] = D_{(U_B)}$ where the test fails to reject H_0 if $D_{\mathbf{0}} \leq D_{(U_B)}$. The OLS full model is asymptotically normal, and hence for large enough n and B the reg len row for the test column should be near $\sqrt{\chi^2_{3,0.95}} = 2.795$.

Were the three values in the test column for reg within 0.1 of 2.795?