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Math 582    HW 9, 2024  Due Wednesday, Nov. 6.
Final: day, TBA: time TBA
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This homework has 2 pages, 3 problems.

1) Suppose that

$$\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{L} N\left(0, \frac{1}{I_1(\mu)}\right).$$

Find a large sample 95% confidence interval for μ.

2) Suppose that $Y_1, ..., Y_n$ are iid from a one parameter exponential family with parameter by τ. Assume that $T_n = \sum_{i=1}^n t(Y_i)$ is a complete sufficient statistics. Suppose, as is often the case, that $T_n \sim G(na, 2b\,\tau)$ where a and b are known positive constants. Then

$$\hat{\tau} = \frac{T_n}{2nab}$$

is the UMVUE and often the MLE of τ. Suggest a $100(1-\alpha)\%$ confidence interval for τ.

Hint: $\dfrac{T_n}{b\tau} \sim G(na, 2)$ and let $P(X \le G(na, 2, \delta/2)) = \delta/2$ and $P(X \le G(na, 2, 1-\delta/2)) = 1 - \delta/2$ if $X \sim G(na, 2)$.

3) Many multiple linear regression estimators $\hat{\boldsymbol{\beta}}$ satisfy

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{D} N_p(0, V(\hat{\boldsymbol{\beta}}, F)\ \boldsymbol{W}) \tag{1}$$

when

$$\frac{\boldsymbol{X}^T\boldsymbol{X}}{n} \xrightarrow{P} \boldsymbol{W}^{-1}, \tag{2}$$

and when the errors e_i are iid with a cdf F and a unimodal pdf f that is symmetric with a unique maximum at 0. When the variance $V(e_i)$ exists,

$$V(OLS, F) = V(e_i) = \sigma^2 \ \text{ while } \ \mathrm{V(L_1, F)} = \frac{1}{\mathrm{4[f(0)]^2}}.$$

In the multiple linear regression model,

$$Y_i = x_{i,1}\beta_1 + x_{i,2}\beta_2 + \cdots + x_{i,p}\beta_p + e_i = \boldsymbol{x}_i^T\boldsymbol{\beta} + e_i \tag{3}$$

for $i = 1, \ldots, n$. In matrix notation, these n equations become

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}, \tag{4}$$

where $\boldsymbol{Y}$ is an $n \times 1$ vector of dependent variables, $\boldsymbol{X}$ is an $n \times p$ matrix of predictors, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown coefficients, and $\boldsymbol{e}$ is an $n \times 1$ vector of unknown errors.

a) What is the ijth element of the matrix

$$\frac{\boldsymbol{X}^T\boldsymbol{X}}{n}?$$

b) Suppose $x_{k,1} = 1$ and that $x_{k,j} \sim X_j$ are iid with $E(X_j) = 0$ and $V(X_j) = 1$ for $k = 1, ..., n$ and $j = 2, ..., p$. Assume that X_i and X_j are independent for $i \neq j$, $i > 1$ and $j > 1$. (Often $x_{k,j} \sim N(0, 1)$ in simulations.) Then what is $\boldsymbol{W}^{-1}$ in (2)?

c) Suppose $p = 2$ and $Y_i = \alpha + \beta X_i + e_i$. Show

$$(\boldsymbol{X}^T \boldsymbol{X})^{-1} = \begin{bmatrix} \frac{\sum X_i^2}{n \sum (X_i - \overline{X})^2} & \frac{-\sum X_i}{n \sum (X_i - \overline{X})^2} \\ \\ \frac{-\sum X_i}{n \sum (X_i - \overline{X})^2} & \frac{n}{n \sum (X_i - \overline{X})^2} \end{bmatrix}$$

d) Under the conditions of c), let $S_x^2 = \sum (X_i - \overline{X})^2 / n$. Show that

$$n(\boldsymbol{X}^T \boldsymbol{X})^{-1} = \left(\frac{\boldsymbol{X}^T \boldsymbol{X}}{n} \right)^{-1} = \begin{bmatrix} \frac{\frac{1}{n} \sum X_i^2}{S_x^2} & \frac{-\overline{X}}{S_x^2} \\ \\ \frac{-\overline{X}}{S_x^2} & \frac{1}{S_x^2} \end{bmatrix}.$$

e) If the X_i are iid with variance $V(X)$ then $n(\boldsymbol{X}^T \boldsymbol{X})^{-1} \xrightarrow{P} \boldsymbol{W}$. What is $\boldsymbol{W}$?

f) Now suppose that n is divisible by 5 and the $n/5$ of X_i are at 0.1, $n/5$ at 0.3, $n/5$ at 0.5, $n/5$ at 0.7 and $n/5$ at 0.9. (Hence if $n = 100$, 20 of the X_i are at 0.1, 0.3, 0.5, 0.7 and 0.9.)

Find $\sum X_i^2 / n$, $\overline{X}$ and S_x^2. (Your answers should not depend on n).

g) Under the conditions of f), estimate $V(\hat{\alpha})$ and $V(\hat{\beta})$ if L_1 is used and if the e_i are iid $N(0, 0.01)$.

Hint: Estimate $\boldsymbol{W}$ with $n(\boldsymbol{X}^T \boldsymbol{X})^{-1}$ and $V(\hat{\boldsymbol{\beta}}, F) = V(L_1, F) = \frac{1}{4[f(0)]^2}$. Hence

$$\begin{pmatrix} \hat{\alpha} \\ \\ \hat{\beta} \end{pmatrix} \approx N_2 \left[\begin{pmatrix} \alpha \\ \\ \beta \end{pmatrix}, \frac{1}{n} \frac{1}{4[f(0)]^2} \begin{pmatrix} \frac{\frac{1}{n} \sum X_i^2}{S_x^2} & \frac{-\overline{X}}{S_x^2} \\ \\ \frac{-\overline{X}}{S_x^2} & \frac{1}{S_x^2} \end{pmatrix} \right]$$

You should get an answer like $0.0648/n$.