To 1h 90 (or 9), added 1 to each score

Math 582 Exam 1, 2022 Name___________

Dm

1) Let $Y_1, ..., Y_n$ be iid double exponential $DE(\theta, \lambda)$ with $E(Y) = \theta$ and $V(Y) = 2\lambda^2$ where θ and y are real and $\lambda > 0$.

a) Find the limiting distribution of $\sqrt{n}[(\overline{Y}) - c]$ for an appropriate constant c. $\sqrt{n}[\overline{Y} - c]$

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$\sqrt{n}(\overline{Y} - \theta) \xrightarrow{D} N(0, 2\lambda^2)$ by CLT

b) Find the limiting distribution of $\sqrt{n}\left[(\overline{Y})^2 - d\right]$ for appropriate constant d for the values of θ where the delta method applies.

$g(\theta) = \theta^2$ $\quad g'(\theta) = 2\theta$, $[g'(\theta)]^2 = 4\theta^2$

$\sqrt{n}[g(\overline{Y}) - \underset{\theta^2}{g(\theta)}] \xrightarrow{D} N[0, [g'(\theta)]^2 \, 2\lambda^2]$

$\sim N(0, 4\theta^2 \, 2\lambda^2) \sim N(0, 8\theta^2\lambda^2)$, $\theta \neq 0$

c) What is the limiting distribution of $n\left[(\overline{Y})^2 - d\right]$ for the value or values of θ where the delta method does not apply? At $\underline{\theta = 0}$ $\quad g'(\theta) = 0$

-1 if not given

$g''(\theta) = 2 > 0$.

So $n[g(\overline{Y}) - g(\theta)] \xrightarrow{D} \frac{1}{2} \, 2\lambda^2 \, g''(\theta) \, \chi^2_1$

or $\boxed{n((\overline{Y})^2 - \underbrace{0}) \xrightarrow{D} 2\lambda^2\chi^2_1}$

$\theta^2 = 0$ if $\theta = 0$

-24

−5 if apply CLT

2) Let $Y_1, ..., Y_n$ be iid beta$(\delta = \theta, \nu = 1)$ with $E(Y) = \frac{\theta}{\theta+1}$, $V(Y) = \frac{\theta}{(\theta+1)^2(\theta+2)}$, and $I_1(\theta) = \frac{1}{\theta^2}$. The beta$(\theta, 1)$ distribution is a 1PREF. Let $\hat{\theta}_n$ be the MLE of θ. Find the limiting distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$. $\xrightarrow{D} N\left(0, \frac{1}{I_1(\theta)}\right)$

$\sim N(0, \theta^2)$

by standard limit thm

(not CLT since $\hat{\theta}_n \neq \bar{Y}$)

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3) Let $Y_1, ..., Y_n$ be iid $C(\mu, \sigma)$. Then the pdf of Y_i is

$$f(y) = \frac{1}{\pi\sigma[1 + (\frac{y-\mu}{\sigma})^2]}$$

where y and μ are real numbers and $\sigma > 0$. Then $MED(Y) = \mu$. Find the limiting distribution of $\sqrt{n}(MED(n) - \mu)$.

$$\sqrt{n}(MED(n) - \mu) \xrightarrow{D} N\left(0, \frac{1}{4[f(\mu)]^2}\right) \sim N\left(0, \frac{\pi^2\sigma^2}{4}\right).$$

$$f(\mu) = \frac{1}{\pi\sigma} \quad \text{so } [f(\mu)]^2 = \frac{1}{\pi^2\sigma^2}$$

$$\text{and } \frac{1}{4[f(\mu)]^2} = \frac{\pi^2\sigma^2}{4} \approx 2.4674\,\sigma^2$$

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$E(Y) = \exp(\mu + \frac{\sigma^2}{2})$ $E(Y^2) = \exp(2\mu + 2\sigma^2)$

4) Let $Y_1, ..., Y_n$ be iid with $E(Y^r) = \exp(r\mu + r^2\sigma^2/2)$ for any real r. Find the limiting distribution of $\sqrt{n}(\overline{Y}_n - c)$ for appropriate constant c.

$$V(Y) = E(Y^2) - (EY)^2 = \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2)$$
$$= e^{2\mu}(e^{2\sigma^2} - e^{\sigma^2}) = e^{2\mu} e^{\sigma^2}(e^{\sigma^2} - 1) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

$$\sqrt{n}\left[\overline{Y} - \exp(\mu + \frac{\sigma^2}{2})\right] \xrightarrow{D} N(0, V(Y))$$

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5) Let $Y_n \sim \text{Poisson}(n\theta)$. Find the limiting distribution of $\sqrt{n}\left(\frac{Y_n}{n} - c\right)$ for appropriate constant c.

$$Y_n = \sum_{i=1}^{n} X_i, \quad X_i \overset{iid}{\sim} Pois(\theta)$$
$$E(X_i) = V(X_i) = \theta$$

$$\therefore \sqrt{n}\left(\frac{Y_n}{n} - \theta\right) \overset{D}{=} \sqrt{n}(\overline{X}_n - \theta) \xrightarrow{D} N(0, \theta)$$

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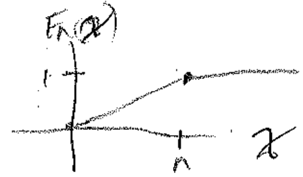


→ 6) Suppose $X_n \sim U(0,n)$. Does $X_n \xrightarrow{D} X$ for some random variable X? Prove or disprove. If $X_n \xrightarrow{D} X$, find X.

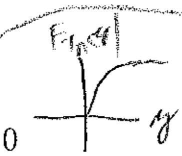
pw

$$F_n(x) = \begin{cases} \frac{x}{n} & 0 < x < n \\ 0 & x \le 0 \end{cases}$$

$\therefore F_n(x) \to 0 \equiv H(x) \quad x \in \mathbb{R}$

$H(x)$ is continuous but not a cdf

$\therefore X_n$ does not converge in dist to any RV X

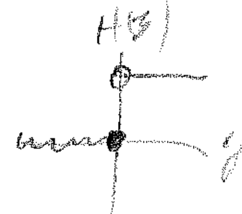


7) Suppose $Y_n \sim EXP(1/n)$ with cdf $F_{Y_n}(y) = 1 - exp(-ny)$ for $y \ge 0$ and $F_{Y_n}(y) = 0$ for $y < 0$. Does $Y_n \xrightarrow{D} Y$ for some random variable Y? Prove or disprove. If $Y_n \xrightarrow{D} Y$, find Y.

$\frac{/}{8}$ pw

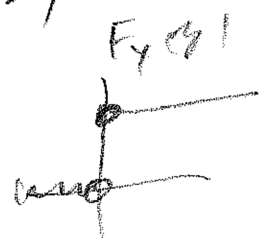
$F_{Y_n}(0) = 0 \ \forall n$ since $e^{-n0} \equiv 1$

$e^{-ny} \to 0 \quad \forall y > 0$



$$\therefore F_{Y_n}(y) \to \begin{cases} 1 & y > 0 \\ 0 & y = 0 \\ 0 & y < 0 \end{cases} = \begin{cases} 1 & y > 0 \\ 0 & y \le 0 \end{cases} = H(y)$$

$\therefore Y_n \xrightarrow{D} Y$ where $P(Y = 0) = 1$



$$F_{Y_n}(y) \to F_Y(y) = \begin{cases} 1 & y \ge 0 \\ 0 & y < 0 \end{cases}$$

except at $y = 0$

$\overline{8}$

8) Suppose $X_1, ..., X_n$ are iid from a distribution with mean μ and variance σ^2. $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} c$. What is c? Hint: Use WLLN on $W_i = X_i^2$.

pm

$$c = E X_i^2 = \sigma^2 + \mu^2.$$

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9) Let $P(X_n = 0) = 1 - 1/n^r$ and $P(X_n = n) = 1/n^r$ where $r > 0$.
a) Prove that X_n does not converge in rth mean to 0. Hint: Find $E[|X_n|^r]$.

pm

$$E[|X_n|^r] = 0 + n^r \frac{1}{n^r} = 1 \quad \forall n$$

$$\text{So } E[|X_n - 0|^r] \to 1 \neq 0.$$

b) Does $X_n \xrightarrow{D} X$ for some random variable X? Prove or disprove. Hint: $P(|X_n - 0| \geq \epsilon) \leq P(X_n = n)$.

$$P[|X_n - 0| \geq \varepsilon] \leq P(X_n = n) = \frac{1}{n^r} \to 0$$

$\therefore X_n \xrightarrow{P} 0 \quad \therefore X_n \xrightarrow{D} 0 \quad \therefore X_n \xrightarrow{D} X$ where $P(X=0)=1$

$$\text{or } F_{X_n}(t) = \begin{cases} 0 & t < 0 \\ 1 - \frac{1}{n^r} & 0 \leq t < n \\ 1 & t \geq n \end{cases} \to F_X(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

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