with solns

Math 582 Exam 2, 2022 Name___________

1) Let $Y_1, ..., Y_n$ be iid $N(\mu, \sigma^2)$. $\theta = \mu$ ←

a) Let $T_{1,n} = \overline{Y}$ and find the limiting distribution of $\sqrt{n}(T_{1,n} - \theta)$.

$$\sqrt{n}\,(T_{1n} - \mu) = \sqrt{n}\,(\overline{Y} - \mu) \xrightarrow{D} N(0, \sigma^2).$$

b) Let $T_{2,n} = \text{MED}(n)$ be the sample median and find the limiting distribution of $\sqrt{n}(T_{2,n} - \theta)$. Hint: $MED(Y) = \mu$.

$$f(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{-1}{2\sigma^2}(y-\mu)^2\right]$$

$$\text{So } f(\mu) = f(MED(Y)) = \frac{1}{\sqrt{2\pi}\,\sigma}$$

$$\sqrt{n}\,(MED(n) - \mu) \xrightarrow{D} N\left(0, \frac{1}{4[f(MED(Y))]^2}\right)$$

$$\sim N\left(0, \frac{2\pi\sigma^2}{4}\right) \sim N\left(0, \frac{\pi\sigma^2}{2}\right).$$

c) Find $ARE(T_{1,n}, T_{2,n})$. Which estimator is better, asymptotically?

$$= \frac{\sigma_2^2(F)}{\sigma_1^2(F)} = \frac{\left(\frac{\pi\sigma^2}{2}\right)}{\sigma^2} = \frac{\pi}{2} = 1.5708$$

So $T_{1n} = \overline{Y}$ is better

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2) Suppose $Y_1, ..., Y_n$ are iid gamma(ν, λ), $Y \sim G(\nu, \lambda)$, where ν is known. Then $I_1(\lambda) = \nu/\lambda^2$. Is $\hat{\lambda}_n = \overline{Y}_n/\nu$ an asymptotically efficient estimator of λ? Hint: determine if

$$\sqrt{n}(\overline{Y}_n/\nu - \lambda) \xrightarrow{D} N\left(0, \frac{1}{I_1(\lambda)}\right).$$

$\sqrt{n}(\overline{Y} - \nu\lambda) \xrightarrow{D} N(0, \nu\lambda^2)$

$\sqrt{n}\,\frac{1}{\nu}(\overline{Y} - \nu\lambda) = \sqrt{n}\left(\frac{\overline{Y}}{\nu} - \lambda\right) \xrightarrow{D} N\left(0, \frac{\lambda^2}{\nu}\right)$

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$\frac{1}{I(\lambda)} = \frac{1}{\nu/\lambda^2} = \frac{\lambda^2}{\nu}$. $\therefore$ $\hat{\lambda}_n$ is asym eff,

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3) Suppose $Y_1, ..., Y_n$ are iid EXP(λ). Let $T_n = Y_{(1)} = Y_{1:n} = \min(Y_1, ..., Y_n)$. It can be shown that the mgf of T_n is

$$m_{T_n}(t) = \frac{1}{1 - \frac{\lambda t}{n}}$$

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for $t < n/\lambda$. Show that $T_n \xrightarrow{D} X$ and give the distribution of X.

$m_{T_n}(t) \to 1 = m_X(t) \quad \forall\, t \in \mathbb{R}$

where $P(X = 0) = 1$,

X is the point mass at 0,

$\sim N(0, 0)$

or $\sim \mathcal{U}$

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4) Suppose $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ are iid 3×1 random vectors from a multinomial distribution with

$$E(\boldsymbol{X}_i) = \begin{bmatrix} m\rho_1 \\ m\rho_2 \\ m\rho_3 \end{bmatrix} \quad \text{and} \quad \text{Cov}(\boldsymbol{X}_i) = \begin{bmatrix} m\rho_1(1-\rho_1) & -m\rho_1\rho_2 & -m\rho_1\rho_3 \\ -m\rho_1\rho_2 & m\rho_2(1-\rho_2) & -m\rho_2\rho_3 \\ -m\rho_1\rho_3 & -m\rho_2\rho_3 & m\rho_3(1-\rho_3) \end{bmatrix}$$

where m is a known positive integer and $0 < \rho_i < 1$ with $\rho_1 + \rho_2 + \rho_3 = 1$. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{X}} - \boldsymbol{c})$ for appropriate vector $\boldsymbol{c}$.

$$\sqrt{n}(\overline{\underset{\sim}{X}} - E(\underset{\sim}{X}_i)) \xrightarrow{D} N_3(\underset{\sim}{0}, \text{Cov}(\underset{\sim}{X}_i)) \quad \text{or}$$

$$\sqrt{n}\left(\overline{\underset{\sim}{X}}_n - \begin{pmatrix} m\rho_1 \\ m\rho_2 \\ m\rho_3 \end{pmatrix}\right) \xrightarrow{D} N_3\left(\underset{\sim}{0}, \begin{bmatrix} m\rho_1(1-\rho_1) & -m\rho_1\rho_2 & -m\rho_1\rho_3 \\ -m\rho_1\rho_2 & m\rho_2(1-\rho_2) & -m\rho_2\rho_3 \\ -m\rho_1\rho_3 & -m\rho_2\rho_3 & m\rho_3(1-\rho_3) \end{bmatrix}\right)$$

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5) Suppose $\boldsymbol{Y}_n \xrightarrow{P} \boldsymbol{Y}$. Then $\boldsymbol{W}_n = \boldsymbol{Y}_n - \boldsymbol{Y} \xrightarrow{P} \boldsymbol{0}$. Define $\boldsymbol{X}_n = \boldsymbol{Y}$ for all n. Then $\boldsymbol{X}_n \xrightarrow{D} \boldsymbol{Y}$. Then $\boldsymbol{Y}_n = \boldsymbol{X}_n + \boldsymbol{W}_n \xrightarrow{D} \boldsymbol{Z}$ by Slutsky's Theorem. What is $\boldsymbol{Z}$?

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$$\underset{\sim}{Y}_n = \underbrace{\underset{\sim}{X}_n}_{\xrightarrow{D} \underset{\sim}{Y}} + \underbrace{\underset{\sim}{W}_n}_{\xrightarrow{P} \underset{\sim}{0}} \xrightarrow{D} \underset{\sim}{Y} + \underset{\sim}{0} = \boxed{\underset{\sim}{Y} = \underset{\sim}{Z}}$$

9 (proves $\underset{\sim}{Y}_n \xrightarrow{D} \underset{\sim}{Y}$ if $\underset{\sim}{Y}_n \xrightarrow{P} \underset{\sim}{Y}$ using Slutsky's thm)

6) If $\boldsymbol{X} \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then the characteristic function of $\boldsymbol{X}$ is

$$c_{\boldsymbol{X}}(\boldsymbol{t}) = \exp\left(i\boldsymbol{t}^T\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{t}^T\boldsymbol{\Sigma}\boldsymbol{t}\right)$$

for $\boldsymbol{t} \in \mathbb{R}^k$. Let $\boldsymbol{a} \in \mathbb{R}^k$ and find the characteristic function of $\boldsymbol{a}^T\boldsymbol{X} = c_{\boldsymbol{a}^T\boldsymbol{X}}(y) = E[\exp(i\ y\ \boldsymbol{a}^T\boldsymbol{X})] = c_{\boldsymbol{X}}(y\boldsymbol{a})$ for any $y \in \mathbb{R}$. Simplify any constants.

$$c_{a^T X}(y) = c_X(y a) = \exp[i\, y a^T \mu - \tfrac{1}{2} y a^T \Sigma a\, y]$$

$$= \exp\left(i\, y\, a^T\mu - \tfrac{1}{2} y^2\, a^T \Sigma a\right)$$

(the chf of a $N(a^T\mu, a^T\Sigma a)$ RV)

9 7) Suppose

$$\sqrt{n}\left(\begin{pmatrix}\hat\theta_1\\ \vdots\\ \hat\theta_p\end{pmatrix} - \begin{pmatrix}\theta_1\\ \vdots\\ \theta_p\end{pmatrix}\right) \xrightarrow{D} N_p(\boldsymbol{0}, \boldsymbol{\Sigma}).$$

Let $\boldsymbol{\theta} = (\theta_1, ..., \theta_p)^T$ and let $\boldsymbol{g}(\boldsymbol{\theta}) = (e^{\theta_1}, ..., e^{\theta_p})^T$. Find $\boldsymbol{D}_{\boldsymbol{g}(\boldsymbol{\theta})}$.

$$D = \begin{pmatrix} \frac{\partial e^{\theta_1}}{\partial\theta_1} & \frac{\partial e^{\theta_1}}{\partial\theta_2} & \cdots & \frac{\partial e^{\theta_1}}{\partial\theta_p} \\ \vdots & & & \\ \frac{\partial e^{\theta_p}}{\partial\theta_1} & \frac{\partial e^{\theta_p}}{\partial\theta_2} & \cdots & \frac{\partial e^{\theta_p}}{\partial\theta_p} \end{pmatrix} = \begin{pmatrix} e^{\theta_1} & & & \\ & e^{\theta_2} & & \\ & & \ddots & \\ & & & e^{\theta_p} \end{pmatrix}$$

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$g(\theta) = e^\theta \to g'(\theta) = e^\theta$

Table 1: Exponential(1) −1 Errors

n	clen	slen	alen	olen	ccov	scov	acov	ocov
50	5.795	6.432	6.821	6.817	.971	.987	.976	.988
100	5.427	5.907	7.525	5.377	.974	.987	.986	.985
1000	5.182	5.387	8.432	4.807	.972	.987	.992	.987
∞	5.152	5.293	8.597	4.605	.972	.990	.995	.990

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8) The above table shows simulation results for multiple linear regression. The large sample 99% PIs are for Y_f given $\boldsymbol{x}_f$ and training data $(Y_1, \boldsymbol{x}_1), ..., (Y_n, \boldsymbol{x}_n)$ with $n = 50, 100$, or 1000. There are 4 PIs s, a, c (classical PI for $N(0, \sigma^2)$ errors, so a Chebyshev PI), and o (asymptotically optimal PI based on the shorth). The distribution for the errors was $EXP(1) - 1$. For each n coverages and the average PI lengths were given. Hence for $n = 50$, PI a had simulated coverage 0.976 and ave. length = 6.821 while $n = 1000$ PI c had simulated coverage 0.972 and ave. length = 5.182. The $n = \infty$ line gives the asymptotic lengths and coverages. There were 5000 runs, so say the PI is best if its coverage ≥ 0.98 with shortest average length. Which PI is best for the following sample sizes n?

a) 50   S   ccov too low
acov too low

b) 100   O   ccov too low

c) 1000   O   ccov too low

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$$\sigma_{ij} = \frac{\alpha_i(1-\alpha_j)}{f(\xi_{\alpha_i})\, f(\xi_{\alpha_j})} \underset{U(0,1)}{=} \alpha_i(1-\alpha_j)$$

f(x) if x ~ U(0,1)

9) Find the limiting distribution of

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$$\sqrt{n}\left((\hat{\xi}_{n,0.75} - \hat{\xi}_{n,0.25}) \;-\; (\xi_{0.75} - \xi_{0.25})\right) = (*)$$

0.25

if the data $Y_1, ..., Y_n$ are iid U(0,1). Then $\xi_\alpha = \alpha$ and $f(\xi_\alpha) = 1$ where $0 < \alpha < 1$.

$$\sqrt{n}\left[\begin{pmatrix}\hat{\xi}_{.25}\\ \hat{\xi}_{.75}\end{pmatrix} - \begin{pmatrix}\xi_{.25}\\ \xi_{.75}\end{pmatrix}\right] \xrightarrow{D} N_2\left[\begin{pmatrix}0\\0\end{pmatrix}, \underbrace{\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{2}\end{pmatrix}}_{\Sigma}\right]$$

$$\sqrt{n}\left(\hat{\xi}_{.75} - \hat{\xi}_{.25} - 0.5\right) \xrightarrow{D} \underline{N(0, \sigma_A^2)} \text{ where}$$

$$\sigma_A^2 = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = \begin{pmatrix}-\sigma_{11} + \sigma_{21}\\ -\sigma_{12} + \sigma_{22}\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix}$$

$$= \sigma_{11} - \sigma_{21} - \sigma_{12} + \sigma_{22} = \sigma_{11} - 2\sigma_{12} + \sigma_{22}$$

$$\sigma_{11} = .25(1-.25) = 0.1875 = \tfrac{3}{16}, \quad \sigma_{12} = \sigma_{21} = .25(1-.75) = \tfrac{1}{16} = 0.0625$$

$$\sigma_{22} = .75(1-.75) = 0.1875 = \tfrac{3}{16}$$

$$\text{So } \sigma_A^2 = 2(0.1875) - 2(.0625) = 2(0.125) = \boxed{0.25}$$

$$= \frac{0.1875}{[f(\xi_{.25})]^2} - \frac{2(0.125)}{f(\xi_{.25})\, f(\xi_{.75})} + \frac{.1875}{[f(\xi_{.75})]^2}$$

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