Math 582 Final, 2022 Name___________

PM

1) Suppose that $X_1, ..., X_n$ are iid $N(\mu, \sigma^2)$.

a) Find the limiting distribution of $\sqrt{n}\ (\overline{X}_n - \mu)$.

$\xrightarrow{D} N(0, \sigma^2)$ by CLT

b) Let $g(\theta) = [\log(1+\theta)]^2$. Find the limiting distribution of $\sqrt{n}\ \left(g(\overline{X}_n) - g(\mu)\right)$ for $\mu > 0$.

$$g'(\theta) = \frac{2\log(1+\theta)}{1+\theta}$$

$$\xrightarrow{D} N(0, [g'(\mu)]^2\sigma^2) = N\left[0, \left(\frac{2\log(1+\mu)}{1+\mu}\right)^2\sigma^2\right]$$

by the delta method

c) Let $g(\theta) = [\log(1+\theta)]^2$. Find the limiting distribution of $n\ \left(g(\overline{X}_n) - g(\mu)\right)$ for $\mu = 0$. Hint: Use the second order delta method.

$g'(0) = 0$ since $\log(1) = 0$, $g''(\theta) = \frac{d}{d\theta}\frac{2\log(1+\theta)}{1+\theta}$

$$= \frac{(1+\theta)\frac{2}{1+\theta} - 2\log(1+\theta)}{(1+\theta)^2} = \frac{2\,[1 - \log(1+\theta)]}{(1+\theta)^2}, \quad g''(0) = \frac{2}{1} = 2$$

Quotient rule

$\left(\frac{n}{d}\right)' = \frac{dn' - nd'}{d^2}$

so $\xrightarrow{D} \frac{1}{2}\sigma^2 2\chi_1^2 = \boxed{\sigma^2\chi_1^2}$

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$Y_n = \sum_{i=1}^{n} X_i$, $E X_i = p$, $V X_i = p(1-p)$

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2) Let $Y_n \sim \text{binomial}(n, p)$.

a) Find the limiting distribution of $\sqrt{n}\left(\frac{Y_n}{n} - p\right)$.

$\xrightarrow{D} N(0, p(1-p))$ by CLT

−9 if n

b) Find the limiting distribution of

$$Z_n = \sqrt{n}\left(\arcsin\left(\sqrt{\frac{Y_n}{n}}\right) - \arcsin(\sqrt{p})\right).$$

$$\text{Hint}: \ \frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

$g'(p) = \frac{1}{\sqrt{1-p^2}}$ or chain rule mistake −6

$g(p) = \arcsin(\sqrt{p})$

$g'(p) = \frac{1}{\sqrt{1-(\sqrt{p})^2}} \frac{1}{2} p^{-\frac{1}{2}} = \frac{1}{2\sqrt{p}\sqrt{1-p}}$

so $[g'(p)]^2 = \frac{1}{4p(1-p)}$

$\therefore Z_n \xrightarrow{D} N[0, p(1-p)[g'(p)]^2]$

$\sim \boxed{N(0, \frac{1}{4})}$

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$(n,1),(-n,0): \quad \frac{1-0}{n-(-n)} = \frac{1}{2n}$

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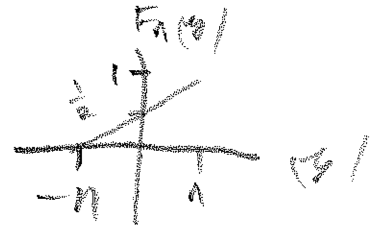
3) Suppose $Y_n \sim \text{uniform}(-n, n)$. Let $F_n(y)$ be the cdf of Y_n.

a) Find $F(y)$ such that $F_n(y) \to F(y)$ for all y as $n \to \infty$.



$F_n(y) = \left(\frac{y}{2n} + \frac{1}{2}\right)$ on $(-n, n)$

$\therefore F_n(y) \to F(y) \equiv \frac{1}{2} \quad \forall y$

b) Does $Y_n \xrightarrow{D} Y$? Explain briefly.

no $F(y)$ is continuous but not a cdf

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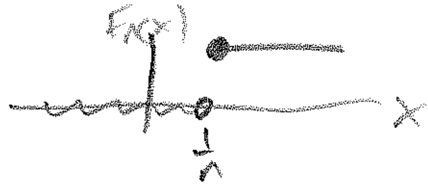
$\left(E(X^2) = \left(\frac{1}{n}\right)^2 \right)$ is wrong work)



4) Let X_n be a sequence of random variables such that $P(X_n = 1/n) = 1$. Does X_n converge in distribution? If yes, prove it by finding X and the cdf of X. If no, prove it.

$X_n \xrightarrow{P} 0$ so $X_n \xrightarrow{D} 0$

cdf easy −10 if X_n does not

or $\lim_{n\to\infty} P(|X_n - 0| < \varepsilon) = 1$ or $E[|X_n - 0|] = EX_n = \frac{1}{n} \to 0$ so $X_n \xrightarrow{P} 0$

or $E(X_n - 0)^2 = 1^2 \frac{1}{n} = \frac{1}{n}$ so $X_n \xrightarrow{2} 0 \Rightarrow X_n \xrightarrow{P} 0$

$P(X=0)=1$

$$F_n(x) = \begin{cases} 1 & x \geq \frac{1}{n} \\ 0 & x < \frac{1}{n} \end{cases} \to \boxed{\begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} = F(x)}$$

$F_n(0) = 0 \ \forall n$

no cdf, −4

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5) Suppose $\boldsymbol{x}_1, ..., \boldsymbol{x}_n$ are iid $p \times 1$ random vectors where $E(\boldsymbol{x}_i) = e^{0.5}\boldsymbol{1}$ and $\text{Cov}(\boldsymbol{x}_i) = (e^2 - e)\boldsymbol{I}_p$. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{x}} - \boldsymbol{c})$ for appropriate vector $\boldsymbol{c}$.

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$$\sqrt{n}\left(\underline{x} - \underbrace{e^{0.5}\, \underset{\sim}{1}}_{\underset{\sim}{c}}\right) \xrightarrow{D} N_p\left(\underline{0},\ (e^2 - e) I_p\right)$$

or 5

by MCLT

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e 6) Assume that

$$\sqrt{n}\left[\begin{pmatrix}\hat\beta_1\\ \hat\beta_2\end{pmatrix} - \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\right] \xrightarrow{D} N_2\left(\underbrace{\begin{pmatrix}0\\ 0\end{pmatrix}}_{\underline{\mu}},\ \underbrace{\begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}}_{\Sigma}\right).$$

Find the limiting distribution of

$$\sqrt{n}[(\hat\beta_1 - \hat\beta_2) - (\beta_1 - \beta_2)] = (1 \quad -1)\sqrt{n}\left[\begin{pmatrix}\hat\beta_1\\ \hat\beta_2\end{pmatrix} - \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\right].$$

$\xrightarrow{D} \boxed{N_1(0, \sigma_1^2+\sigma_2^2)}$

$\underline{a}\,\underline{\mu} = \underline{a}\,\underline{0} = 0$, $\underline{a}\,\Sigma\,\underline{a}^T = (1 \ -1)\begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}$

$= (\sigma_1^2 \ -\sigma_2^2)\begin{pmatrix}1\\ -1\end{pmatrix} = \sigma_1^2+\sigma_2^2$

17 e 7) Let $X_1, ..., X_n$ be iid with mean $E(X) = \mu$ and variance $V(X) = \sigma^2 > 0$. Then $n(\overline{X} - \mu)^2 = [\sqrt{n}(\overline{X} - \mu)]^2 \xrightarrow{D} W$. What is W?

$\sqrt{n}(\bar X - \mu) \xrightarrow{D} Z \sim N(0, \sigma^2) \sim \sigma N(0,1)$ by CLT

So $W = Z^2 \sim \sigma^2 \chi^2_1$ by

continuous mapping Th

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8) Suppose that $Y_1, ..., Y_n$ are iid logistic$(\theta, 1)$ with pdf

$$f(y) = \frac{\exp(-(y-\theta))}{[1+\exp(-(y-\theta))]^2}$$

where ~~and~~ y and θ are real. MED$(Y) = E(Y) = \theta$ and $V(Y) = \pi^2/3$.

e a) $I_1(\theta) = 1/3$ and the family is regular so the "standard limit theorem" for the MLE $\hat{\theta}_n$ holds. Using this standard theorem, what is the limiting distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$?

$\xrightarrow{D} N\left(0, \frac{1}{I_1(\theta)}\right) \sim \boxed{N(0,3)}$

e b) Find the limiting distribution of $\sqrt{n}(\overline{Y}_n - \theta)$.

$\xrightarrow{D} N\left(0, \underbrace{\frac{\pi^2}{3}}_{\approx 3.2899}\right)$ by CLT

c) Find the limiting distribution of $\underbrace{\sqrt{n}(MED(n) - \theta)}_{}$. $z_n = \xrightarrow{D} N\left(0, \frac{1}{4[f(\theta)]^2}\right)$

$f(\theta) = \frac{1}{(1+1)^2} = \frac{1}{4}$, $4[f(\theta)]^2 = \frac{4}{16} = \frac{1}{4}$

$z_n \xrightarrow{D} \boxed{N(0,4)}$

d) Consider the estimators $\hat{\theta}_n$, $\overline{Y}_n$ and MED(n). Which is the best estimator and which is the worst?

best smallest asy var

worst, largest asy var

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9) Let the linear model $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{e}$ where $\boldsymbol{X}$ has full rank p, $E(\boldsymbol{e}) = \boldsymbol{0}$, and $\text{Cov}(\boldsymbol{e}) = \sigma^2 \boldsymbol{I}$. Let $\hat{\boldsymbol{\eta}}_A$ be the OLS, lasso, elastic net, or ridge regression estimator fit from the model $\boldsymbol{Z} = \boldsymbol{W\eta} + \boldsymbol{e}$. If $\hat{\lambda}_{1,n}/\sqrt{n} \xrightarrow{P} 0$, then $\sqrt{n}(\hat{\boldsymbol{\eta}}_A - \cancel{\boldsymbol{\beta}}\ \boldsymbol{\eta}) \xrightarrow{D} \boldsymbol{u}$ for all 4 estimators.

a) What is the distribution of $\boldsymbol{u}$?

$\underline{u} \sim N_{p-1}(\underline{0}, \sigma^2 V)$

b) Let $\boldsymbol{a}$ be a $(p-1) \times 1$ ~~$p \times 1$~~ constant vector. Then for a large class of iid error distributions, what is the limiting distribution of $\boldsymbol{a}^T \sqrt{n}(\hat{\boldsymbol{\eta}}_A - \boldsymbol{\eta}) = \sqrt{n}(\boldsymbol{a}^T \hat{\boldsymbol{\eta}}_A - \boldsymbol{a}^T \boldsymbol{\eta})$?

$\xrightarrow{D} N_1(\underline{0}, \sigma^2\, \underline{a}^T V \underline{a})$

p-1 or p-s

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