M582 010 Q1

Math 595 Quiz 2 Fall 2005 Name___________

1) Suppose that $Y_1, ..., Y_n$ are iid with $E(Y) = (1-\rho)/\rho$ and $\text{VAR}(Y) = (1-\rho)/\rho^2$ where $0 < \rho < 1$. a) Find the limiting distribution of $\sqrt{n}\left(\overline{Y}_n - \frac{1-\rho}{\rho}\right)$.

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5.31 HW 8.17

$\xrightarrow{L} N\left(0, \frac{1-\rho}{\rho^2}\right)$ by the CLT.

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b) Find the limiting distribution of $\sqrt{n}\left[g(\overline{Y}_n) - \rho\right]$ for appropriate function g.

Want $g(\mu)$ so that $g\left(\frac{1-\rho}{\rho}\right) = \rho$. Now $\frac{1-\rho}{\rho} = \frac{1}{\rho} - 1$

So $g(\mu) = \frac{1}{1+\mu}$. Note that $g\left(\frac{1-\rho}{\rho}\right) = \frac{1}{1+\frac{1-\rho}{\rho}} \frac{\rho}{\rho} = \frac{\rho}{\rho+1-\rho} = \rho$

By the delta method, $\sqrt{n}\left[g(\overline{Y}) - g(\mu)\right] \xrightarrow{L} N(0, (g'(\mu))^2 \tau^2)$

if $\sqrt{n}(\overline{Y} - \mu) \xrightarrow{L} N(0, \tau^2)$. Here $\mu = \frac{1-\rho}{\rho}$

$g(\mu) = \frac{1}{1+\mu}$, $g'(\mu) = \frac{-1}{(1+\mu)^2}$ and $[g'(\mu)]^2 = \frac{1}{(1+\mu)^4}$

So $\left[g'\left(\frac{1-\rho}{\rho}\right)\right]^2 = \frac{1}{\left(1+\frac{1-\rho}{\rho}\right)^4} = \frac{1}{(1+\frac{1}{\rho}-1)^4} = \rho^4$

So $\sqrt{n}\left[g(\overline{Y}) - \rho\right] \xrightarrow{L} N\left(0, \frac{\rho^4}{\rho^2}(1-\rho)\right) = N\left(0, \rho^2(1-\rho)\right)$

$\to N\left[0, \frac{1-\rho}{\rho^2}\left[g'\left(\frac{1-\rho}{\rho}\right)\right]^2\right]$ 1

See 2.12

M582 old Q1

Math 595 Quiz 3 Fall 2005 Name___________

1) Let $X_n \sim \text{Binomial}(n, p)$ where the positive integer n is large and $0 < p < 1$.

33 a) Find the limiting distribution of $\sqrt{n}\left(\frac{X_n}{n} - p\right)$.

$X_n \stackrel{D}{=} \sum_{i=1}^{n} Y_i$ where $Y_i \sim \text{Bin}(1,p)$, $EY_i = p$ $\text{Var}\, Y_i = p(1-p)$

So by CLT $\sqrt{n}\left(\frac{X_n}{n} - p\right) \xrightarrow{L} N(0, p(1-p))$.

34 b) Find the limiting distribution of $\sqrt{n}\left[\left(\frac{X_n}{n}\right)^2 - p^2\right]$.

Let $g(p) = p^2$. Then $g'(p) = 2p$ and by the delta method $\sqrt{n}\left(\frac{X_n}{n} - p^2\right) \xrightarrow{L} N[0, p(1-p)\underbrace{(g'(p))^2}_{4p^2}]$

$= N[0, 4p^3(1-p)]$.

34 c) Let $g(\theta) = \theta^3 - \theta$. Find the limiting distribution of $n\left[g\left(\frac{X_n}{n}\right) - c\right]$ for appropriate constant c when $p = \frac{1}{\sqrt{3}}$. Hint: See (2.5.14).

If $g'(\theta) = 0$, $g''(\theta) \neq 0$ and $\sqrt{n}(T_n - \theta) \xrightarrow{L} N(0, \tau^2(\theta))$

then $n[g(T_n) - g(\theta)] \xrightarrow{L} \frac{1}{2}\tau^2(\theta)\, g''(\theta)\, \chi^2_1$.

Now $g'(\theta) = 3\theta^2 - 1$, $g''(\theta) = 6\theta$

$g\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \frac{-2}{3\sqrt{3}}$, $g'\left(\frac{1}{\sqrt{3}}\right) = 0$ $g''\left(\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}}$

$\tau^2(p) = p(1-p)$ by a). So $\tau^2\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{\sqrt{3}}\right)$.

So $n\left[g\left(\frac{X_n}{n}\right) - \underbrace{\left(\frac{-2}{3\sqrt{3}}\right)}_{c}\right] \xrightarrow{L} \frac{1}{2}\frac{1}{\sqrt{3}}\left(1 - \frac{1}{\sqrt{3}}\right)\frac{6}{\sqrt{3}}\chi^2_1$

$= \left(1 - \frac{1}{\sqrt{3}}\right)\chi^2_1$

$\frac{1}{2}\tau^2(p)\, g''(p) = \frac{1}{2}p(1-p)6p = 3p^2(1-p) = 3\left(\frac{1}{\sqrt{3}}\right)^2\left(1 - \frac{1}{\sqrt{3}}\right) = 1 - \frac{1}{\sqrt{3}}$