Math 582 Quiz 10 2022 Name__________

1) Suppose $\boldsymbol{x}_1, ..., \boldsymbol{x}_n$ are iid $p \times 1$ random vectors where

$$\boldsymbol{x}_i \sim (1-\gamma) N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) + \gamma N_p(\boldsymbol{\mu}, c\boldsymbol{\Sigma})$$

with $0 < \gamma < 1$ and $c > 0$. Then $E(\boldsymbol{x}_i) = \boldsymbol{\mu}$ and $\text{Cov}(\boldsymbol{x}_i) = [1 + \gamma(c-1)]\boldsymbol{\Sigma}$. Find the limiting distribution of $\sqrt{n}(\overline{\boldsymbol{x}} - \boldsymbol{d})$ for appropriate vector $\boldsymbol{d}$.

$$\sqrt{n}(\overline{x} - \mu) \xrightarrow{D} N_p(\underline{0}, [1 + \gamma(c-1)]\Sigma)$$

by the MCLT

2) Suppose $\boldsymbol{Y} \sim N_n(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{I})$. Find the distribution of $\boldsymbol{HY}$ if $\boldsymbol{H}$ is an $n \times n$ constant matrix such that $\boldsymbol{HX} = \boldsymbol{X}$ and $\boldsymbol{H} = \boldsymbol{H}^T = \boldsymbol{HH} = \boldsymbol{H}^2$. Simplify.

$$HY \sim N_n(HX\beta, \sigma^2 HH^T)$$

$$\sim N_n(X\beta, \sigma^2 H)$$

3) Suppose that $Y_i = \alpha + \boldsymbol{x}_i^T \boldsymbol{\beta} + e_i$ where the $e_i = \sigma_i \epsilon_i$ where the ϵ_i iid with $E(\epsilon_i) = 0$ and $V(\epsilon_i) = 1$. Then the e_i are independent with $E(e_i) = 0$ and $V(e_i) = \sigma_i^2$. This MLR model can be written as $\boldsymbol{Y} = \alpha \boldsymbol{1} + \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}$. We will assume that the cases $(\boldsymbol{x}_i^T, Y_i)^T$ are iid. Fit the model with OLS to get $(\hat{\alpha}, \hat{\boldsymbol{\beta}})$ and the residuals r_i. The nonparametric bootstrap samples the $(\boldsymbol{x}_i, Y_i, r_i)$ with replacement to form the MLR model $\boldsymbol{Y}^* = \hat{\alpha}\boldsymbol{1} + \boldsymbol{X}^*\hat{\boldsymbol{\beta}} + \boldsymbol{r}^*$ where with respect to the bootstrap distribution, the r_i^* are iid with $E(r_i^*) = 0$. This bootstrap model has the $(\boldsymbol{x}_i^{*T}, Y_i)^T$ iid with respect to the bootstrap distribution.

The MLR model $\boldsymbol{Y}^* = \hat{\alpha}\boldsymbol{1} + \boldsymbol{X}^*\hat{\boldsymbol{\beta}} + \boldsymbol{r}^*$ is the bootstrap data set, and OLS is fit to the model to obtain the bootstrapped statistic $(\hat{\alpha}^* = \overline{Y^*} - \hat{\boldsymbol{\beta}}^{*T}\overline{\boldsymbol{x}^*}, \hat{\boldsymbol{\beta}}^* = \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}^*}^{-1}\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}^*Y^*})$.

a) By the second method to compute OLS, $\hat{\boldsymbol{\beta}} = \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}}^{-1}\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}Y}$. Since the bootstrap distribution for the nonparametric bootstrap is the empirical distribution, it can be shown that $[\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}}^*]^{-1} \xrightarrow{P} \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}}^{-1}$ and $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}Y}^* \xrightarrow{P} \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}Y}$. Prove that $\hat{\boldsymbol{\beta}}^* = [\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}}^*]^{-1}\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{x}Y}^* \xrightarrow{P} \hat{\boldsymbol{\beta}}$.

$$[\tilde{\Sigma}_{\underline{x}}^*]^{-1} \tilde{\Sigma}_{\underline{xy}}^* \xrightarrow{P} \tilde{\Sigma}_{\underline{x}}^{-1} \tilde{\Sigma}_{\underline{xy}} = \hat{\underline{\beta}}$$

b) By the second method to compute OLS, $\hat{\alpha} = \overline{Y} - \hat{\boldsymbol{\beta}}^T\overline{\boldsymbol{x}}$. It can be shown that $\overline{Y^*} \xrightarrow{P} \overline{Y}$ and $\overline{\boldsymbol{x}^*} \xrightarrow{P} \overline{\boldsymbol{x}}$. Prove that $\hat{\alpha}^* = \overline{Y^*} - \hat{\boldsymbol{\beta}}^{*T}\overline{\boldsymbol{x}^*} \xrightarrow{P} \hat{\alpha}$.

$$\overline{Y}^* - \hat{\underline{\beta}}^{*T} \overline{\underline{X}}^* \xrightarrow{P} \overline{Y} - \hat{\underline{\beta}}^T \overline{\underline{X}} = \hat{\alpha}.$$

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