Math 582 Quiz 11 2022 Name___________

1) Suppose $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{e}$ satisfies the OLS CLT. Let $r_i = Y_i - \hat{Y}_i$ be the ith OLS residual for $i = 1, ..., n$ where a constant is in the model. The residual bootstrap draws the residuals with replacement to form the model $\boldsymbol{Y}^* = \boldsymbol{X}\hat{\boldsymbol{\beta}} + \boldsymbol{r}^*$. X is $n \times p$.

Suppose that for some large fixed m, OLS is fit to find $\hat{\boldsymbol{\beta}}_m$ and the m OLS residuals. Then as $n \to \infty$, the m residuals are drawn with replacement to form $\boldsymbol{Y}^* = \boldsymbol{X}\hat{\boldsymbol{\beta}}_m + \boldsymbol{r}^*$ where $Y_i^* = \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}_m + r_i^*$ for $i = 1, ..., n$. This model satisfies the OLS CLT with the r_i^* iid with respect to the bootstrap distribution, $E(r_i^*) = 0$, and $V(r_i^*) = \sigma_m^2 = \frac{1}{m}\sum_{i=1}^m r_i^2 = \frac{m-1}{m} MSE(m)$ where $MSE(m) \xrightarrow{P} \sigma^2 = V(e_i)$ as $m \to \infty$. The bootstrap estimator $\hat{\boldsymbol{\beta}}^*$ is found by fitting OLS to the model $\boldsymbol{Y}^* = \boldsymbol{X}\hat{\boldsymbol{\beta}}_m + \boldsymbol{r}^*$. By the OLS CLT,

$$\sqrt{n}(\hat{\boldsymbol{\beta}}^* - \hat{\boldsymbol{\beta}}_m) \xrightarrow{D} \boldsymbol{u}.$$

What is the distribution of $\boldsymbol{u}$?

$$\sim N_p(\underline{0}, \sigma_m^2 W)$$

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2) Show that $\sum_{i=1}^n (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(Y_i - \overline{Y}) = \sum_{i=1}^n (\boldsymbol{x}_i - \overline{\boldsymbol{x}}) Y_i$.

$$\overline{Y}\sum_{i=1}^n (X_i - \overline{X}) = \overline{Y}\left(\sum_{i=1}^n \underline{X}_i - n\overline{X}\right) = \overline{Y}(n\underline{\overline{X}} - n\underline{\overline{X}}) = 0.$$

$$= \overline{Y}\left(\sum X_i - n\overline{X}\right)$$

$$= \overline{Y} n\overline{X} - n\overline{X}\overline{Y}$$

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3) Let $\boldsymbol{\Sigma}_i$ be the nonsingular population covariance matrix of the ith treatment group or population. To simplify the large sample theory, assume $n_i = \pi_i n$ where $0 < \pi_i < 1$ and $\sum_{i=1}^3 \pi_i = 1$. Let T_i be a multivariate location estimator such that $\sqrt{n_i}(T_i - \boldsymbol{\mu}_i) \xrightarrow{D} N_m(\mathbf{0}, \boldsymbol{\Sigma}_i)$, and $\sqrt{n}(T_i - \boldsymbol{\mu}_i) \xrightarrow{D} N_m\left(\mathbf{0}, \frac{\boldsymbol{\Sigma}_i}{\pi_i}\right)$ for $i = 1, 2, 3$. Assume the T_i are independent.

Then

$$\sqrt{n}\begin{bmatrix} T_1 - \boldsymbol{\mu}_1 \\ T_2 - \boldsymbol{\mu}_2 \\ T_3 - \boldsymbol{\mu}_3 \end{bmatrix} \xrightarrow{D} \boldsymbol{u}.$$

a) Find the distribution of $\boldsymbol{u}$.

$$N_{3m}\left[\begin{pmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \\ \underset{\sim}{0} \end{pmatrix}, \begin{pmatrix} \frac{\Sigma_1}{\pi_1} & 0 & 0 \\ 0 & \frac{\Sigma_2}{\pi_2} & 0 \\ 0 & 0 & \frac{\Sigma_3}{\pi_3} \end{pmatrix}\right]$$

b) Suggest an estimator $\hat{\pi}_i$ of π_i.

$$\hat{\pi}_i = \frac{n_i}{n}$$

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